

# On the lattice of normal subgroups in ultraproducts of compact simple groups

Abel Stolz      Andreas Thom

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## Abstract

We prove that the lattice of normal subgroups of ultraproducts of compact simple non-abelian groups is distributive. In the case of ultraproducts of finite simple groups or compact connected simple Lie groups of bounded rank the set of normal subgroups is shown to be linearly ordered by inclusion.

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# 1 Introduction

This article is about the structure of the lattice of normal subgroups of an ultraproduct of compact simple groups. Note that the Peter-Weyl theorem implies that any compact simple group is either finite simple or a finite-dimensional connected compact simple Lie group; both cases admit a complete classification which is the basis of our considerations.

The motivation to study ultraproducts of groups is manifold. First of all, *qualitative* properties of the ultraproduct reflect *quantitative* properties of the groups involved. This becomes interesting since the manipulation of qualitative properties is sometimes easier and can be done with geometric or algebraic insight which might not be available in the quantitative computations. Let us give an example: We will later show that the set of normal subgroups of an ultraproduct of non-abelian finite simple groups is linearly ordered. Equivalently, there exists a natural number  $k$ , such that for each non-abelian simple group  $G$  and each  $g, h \in G$  either  $g \in (C(h) \cup C(h^{-1}))^k$  or  $h \in (C(g) \cup C(g^{-1}))^k$ , where  $C(s)$  denotes the conjugacy class of some element  $s \in G$ . While the qualitative statement sounds natural, the quantitative statement looks a bit more surprising at first. These statements can be proved only using the classification of finite simple groups.

Another source of motivation for the study of normal subgroups of ultraproducts is the recent interest in sofic groups which has lead to the consideration of metric ultraproducts of symmetric groups. The connection between the two topics is a theorem of Elek and Szabó [6], asserting that a countable group is sofic if and only if embeds into a metric ultraproduct of symmetric groups. A *metric* ultraproduct of groups is the quotient of an ultraproduct  $\mathbf{G}$  by a normal subgroup  $\mathbf{N}$ , arising as the set of all elements infinitesimally close to the identity. In this context distance is measured in the Hamming distance on permutation groups. In contrast to the theory of sofic groups, where the above subgroup  $\mathbf{N}$  is neglected, Ellis et al. [7] investigated this very normal subgroup, starting with the observation that it is maximal and hence  $\mathbf{G}/\mathbf{N}$  simple. In fact they were able to show that the normal subgroups of  $\mathbf{G}$  are linearly ordered by inclusion. Thus naturally the problem arose whether this theorem would generalize to ultraproducts of other (possibly all) non-abelian finite simple groups. We answer this question in the positive. The main source of knowledge used in the proof is Liebeck and Shalev's deep investigation of the size of conjugacy classes in finite simple groups [11]. Having thus dealt with all finite simple groups, where one can hope a priori for a positive answer, we take one more step to compact simple groups. In this setting an analogous theorem still holds true, under the somewhat restricting assumption of imposing a bound on the rank of groups contributing to the ultraproduct in question. When the bound on the rank is dropped the lattice of normal subgroups is no longer linearly ordered but we can still show that it is distributive, and in fact not very complicated. The method of proof takes its inspiration from seminal work of Nikolov and Segal [13].

The article is organized as follows. Section 2 introduces notation and basic notions of metrics on permutation groups and matrix groups. Quite some effort is dedicated to the study of the connection of the Hamming distance to the size

of conjugacy classes in symmetric groups. Although these results are not used in the sequel, they illucidate the above mentioned theorem of Ellis et al. In fact one could easily reprove the theorem combining the results in Section 2 and Section 3.

Section 3 starts with some geometric prerequisites and culminates in Theorem 3.9, validating the claim that the set of normal subgroups of ultraproducts of non-abelian finite simple groups is linearly ordered.

Section 4 consists mainly of an investigation of the inner structure of compact connected simple Lie groups. Then Theorem 4.11 for ultraproducts of compact connected simple Lie groups with bounded rank is deduced. Considering Lie groups of unbounded rank leads to Theorem 4.20, asserting that the lattice of normal subgroups of ultraproducts of these is distributive.

In the concluding section the obtained results are bundled into Main Theorem 5.1.

The reader is assumed to be familiar with ultraproducts and ultralimits. Elementary properties of ultrafilters will be used without further notice. For a comprehensive introduction to (metric) ultraproducts and related notions confer [3]. We use more facts concerning finite simple groups of Lie type and Lie groups, respectively, than we are willing to introduce thoroughly. One may use the textbooks cited below or some standard reference of one's own choice, to verify missing links. Note that – from now on – when talking about finite simple group we always mean finite simple *non-abelian* groups.

## 2 Length functions

The study of groups is enriched when we introduce a compatible metric or topology. Metric or topological groups are a well understood subject of study. In this section we will introduce the notion of pseudo length functions, which in fact is just a reformulation of the notion of pseudometrics. We shall further give examples of pseudo length functions in some, mostly finite, groups and examine how different pseudo length functions can be compared in large groups.

We denote the set  $\{1, \dots, n\}$  of natural numbers by  $[n]$ . In a group  $G$  we write  $g^G$  for the conjugacy class of an element  $g$ . The group generated by  $g$  is  $\langle g \rangle$ , the group generated by a subset  $S \subset G$  is  $\langle S \rangle$ , and consequently the normal subgroup generated by  $g$  is  $\langle g^G \rangle$ . When the group in which conjugation takes place is understood, we write  $C(g)$  for the conjugacy class of  $g$  and  $N(g)$  for the normal closure of  $g$ . We call  $S$  normal if it is the union of conjugacy classes and non-trivial if it contains a non-identity element. For a natural number  $n$ , the elementwise power of  $X \subset G$  is  $X^{\bullet n} := \{g^n \mid g \in X\}$ .

Let  $G$  be a group. A function  $\ell : G \rightarrow [0, \infty[$  is called a **length function** on  $G$  if for all  $g, h \in G$

LF1  $\ell(g) = 0$  if and only if  $g = 1$ ,

LF2  $\ell(g) = \ell(g^{-1})$ ,

LF3  $\ell(gh) \leq \ell(g) + \ell(h)$ .

If moreover  $\ell(hgh^{-1}) = \ell(g)$  holds, then  $\ell$  is **invariant**. If the first axiom is weakened to  $\ell(1) = 0$ , then  $\ell$  is only a **pseudo length function**.

It is an easy observation that every (pseudo) length function corresponds to a right-invariant (pseudo) metric on  $G$  and vice versa by  $d(g, h) = \ell(gh^{-1})$  and  $\ell(g) = d(g, 1)$ . The notion of invariance for (pseudo) length functions translates into left-invariance of the corresponding (pseudo) metric. We say that a group  $G$  has **diameter**  $D(G)$  with respect to  $\ell$  if  $\sup_{g \in G} \ell(g) = D(G)$ . This notion coincides with the diameter of metric spaces.

It will turn out to be necessary to study the asymptotics of sequences of pseudo length functions on groups of growing size. Let  $\mathcal{G} = \{G_n \mid n \in \mathbb{N}\}$  be a countably infinite family of groups with generic length functions  $\ell_1$  and  $\ell_2$  defined for every  $G \in \mathcal{G}$ . We call  $\ell_1$  **asymptotically bounded** by  $\ell_2$  if there are constants  $c$  and  $N$  such that for every  $n \geq N$  and every choice of elements  $g \in G_n$  we have  $\ell_1(g) \leq c\ell_2(g)$ . The constant  $c$  is called a **modulus** of asymptotic boundedness. The function  $\ell_1$  is **locally asymptotically bounded** by  $\ell_2$  in **radius**  $\delta$ , if the same holds for all  $g \in G_n$  satisfying  $\ell_1(g) < \delta$ , for some  $\delta > 0$  not depending on  $n$ . We call  $\ell_1$  and  $\ell_2$  **(locally) asymptotically equivalent** if  $\ell_1$  and  $\ell_2$  are (locally) asymptotically bounded with respect to each other.

We are interested in the interaction of pseudo length functions and quotient groups. The following two lemmas introduce the natural definitions.

**2.1 Lemma** *Let  $G$  be a finite group with a normal subgroup  $H$  and an invariant (pseudo) length function  $\ell$ . Then*

$$\ell_{G/H}(gH) := \inf_{h \in H} \ell(gh)$$

*defines an invariant (pseudo) length function on  $G/H$ .*

**Proof.** We only show the triangle inequality. Let  $g, h$  be in  $G$  and  $k, l$  in  $H$  such that  $\ell(gk)$  and  $\ell(hl)$  are minimal. Then

$$\ell_{G/H}(ghH) \leq \ell(gkhl) \leq \ell(gk) + \ell(hl) = \ell_{G/H}(gH) + \ell_{G/H}(hH). \quad \square$$

The proof of the following statement is obvious.

**2.2 Lemma** *Let  $G$  be a group with normal subgroup  $H$  and  $\ell$  a (pseudo) length function on  $G/H$ . Then*

$$\ell^G(g) := \ell(gH)$$

*defines a (pseudo) length function on  $G$ . If  $\ell$  is invariant, then  $\ell^G$  is invariant, too.*

## 2.1 The conjugacy length

An example of a pseudo length function that can be defined on any finite group  $G$  is the **conjugacy length**

$$\ell_c(g) := \frac{\log |C(g)|}{\log |G|}.$$

**2.3 Proposition** *Let  $G$  be a finite group. Then the function  $\ell_c$  is an invariant pseudo length function on  $G$ .*

*Proof.* Obviously  $\ell_c$  only takes values in the interval  $[0, 1]$ . The conjugacy class of 1 has one element, conjugacy classes of mutually inverse elements have the same size, and conjugacy classes of products are contained in the respective product of conjugacy classes. Therefore  $\ell_c$  is a pseudo length function. By definition it is invariant under conjugation.  $\square$

Note that  $\ell_c$  is a length function if and only if  $G$  has trivial center, in particular if  $G$  is non-abelian and simple. More explicit is the following proposition.

**2.4 Proposition** *Let  $G$  be a finite group. Then*

$$\ell_c(g) = \ell_{c_{G/Z(G)}}(gZ(G))$$

*holds for all  $g \in G$ .*

*Proof.* It is not hard to observe that  $|C(gz)| = |C(g)|$  for any central element  $z$ , which proves

$$\ell_{c_{G/Z(G)}}(gZ(G)) = \inf_{z \in Z(G)} \ell_c(gz) = \inf_{z \in Z(G)} \ell_c(g) = \ell_c(g). \quad \square$$

**2.5 Lemma** *Let  $G$  be a finite group. Then for all  $g \in G$  and  $n \in \mathbb{N}$  the estimate*

$$\ell_c(g^n) \leq \ell_c(g)$$

*holds.*

*Proof.* Let  $h$  be an arbitrary element in the conjugacy class of  $g^n$ , say  $h = xg^nx^{-1}$ . Then  $h = (xgx^{-1})^n \in C(g)^{\bullet n}$ . Because  $|C(g)^{\bullet n}| \leq |C(g)|$ , the claim follows.  $\square$

The conjugacy length is very useful, because it is directly related to algebraic properties of the group. We will make heavy use of results of Liebeck-Shalev. They showed in [11] that in any non-abelian simple group  $G$ , a conjugacy class of some element generates  $G$  essentially as quickly as the conjugacy length permits. More precisely, Liebeck-Shalev showed that there is a constant  $k$ , such that  $C(g)^{k/\ell_c(g)} = G$  for all non-abelian finite simple groups  $G$  and all  $g \in G$ . On the other side it is clear that at least  $D(G)/\ell_c(g)$  products are necessary, and  $D(G)$  is bounded below by a positive constant. Hence, the result of Liebeck-Shalev is best possible. One drawback is that the conjugacy length is not directly related to geometry and sometimes hard to compute. We will proceed by giving some examples of length functions on classes of groups from everyday life and show that for each finite simple group the conjugacy length can be replaced by more familiar invariant length functions related to geometry.

## 2.2 Length functions on permutation groups

We denote the class of all symmetric groups (i.e. full permutation groups of finite sets) by  $\mathcal{S}$  and the class of alternating groups by  $\mathcal{A}$ .

Length functions necessarily have to be constant on conjugacy classes. Therefore the following statement sounds reasonable.

**2.6 Proposition** *Let  $\pi$  be a permutation in  $S_n$  with  $l$  cycles. Then*

$$\ell_r(\pi) := 1 - \frac{l}{n}$$

*defines an invariant length function on  $S_n$ .*

We postpone the proof to Subsection 2.3 and look at another example.

The **Hamming length** of a permutation  $\pi \in S_n$  is defined as

$$\ell_H(\pi) := 1 - \frac{|\{i \in [n] \mid \pi(i) = i\}|}{n}$$

It is well known that  $\ell_H$  is an invariant length function on  $S_n$ .

The following proposition serves as an introductory example of asymptotic equivalence and will be useful later.

**2.7 Proposition** *The length functions  $\ell_H$  and  $\ell_r$  are asymptotically equivalent.*

**Proof.** Let  $\pi \in S_n$  with  $l$  cycles,  $m$  of which are trivial. Then immediately

$$\ell_r(\pi) = \frac{n-l}{n} \leq \frac{n-m}{n} = \ell_H(\pi)$$

follows. Because the remaining  $l-m$  non-trivial cycles have length at least 2,  $l-m \leq \frac{1}{2}(n-m)$ . We conclude

$$\frac{n-m}{n} = \frac{n-l+l-m}{n} \leq \frac{n-l}{n} + \frac{n-m}{2n}$$

and finally  $\ell_H(\pi) \leq 2\ell_r(\pi)$ .  $\square$

In this paragraph we use the notation  $\gamma(x) := \Gamma(x+1)$ , hence  $\gamma(k) = k!$  for natural numbers  $k$ . (The traditional symbol, introduced by Gauss, is  $\Pi$  instead of  $\gamma$ .) There is an extensive list of known inequalities of the  $\Gamma$ -function; however we could not locate the following result in the literature. We will need the following proposition later.

**2.8 Proposition** *For all  $x, y \geq 0$  the inequality  $\gamma(x)\gamma(y) \leq \gamma(x+y)$  holds.*

**Proof.** We use the characterization of the Gamma function, introduced by Euler, as the limit

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n^x n!}{x(x+1) \dots (x+n)},$$

for all  $x, y > 0$ . Let  $k \geq 1$  and  $x, y \geq 0$ . Then

$$(x+k)(y+k) = k(x+y) + k^2 + xy \geq k(x+y+k)$$

and consequently also

$$\begin{aligned} & \frac{n^{x+1} n^{y+1} (n!)^2}{(x+1)(y+1) \dots (x+n+1)(y+n+1)} \\ & \leq \frac{n \cdot n! \cdot n^{x+y+1} n!}{(n+1)!(x+y+1) \dots (x+y+n+1)} \\ & \leq \frac{n^{x+y+1} n!}{(x+y+1) \dots (x+y+n+1)}. \end{aligned}$$

In the limit we obtain the desired inequality

$$\gamma(x)\gamma(y) = \Gamma(x+1)\Gamma(y+1) \leq \Gamma(x+y+1) = \gamma(x+y). \quad \square$$

We shall use the remainder of this paragraph to exhibit the connection of the Hamming length to the generic conjugacy length introduced above. To this end we must go through some unpleasant and tedious estimates. Note that the next results do only hold in permutation groups of considerable size, although we refrain from giving explicit lower bounds assuring validity of the statements to prove.

**2.9 Lemma** *Let  $\pi$  be a permutation in  $S_n$ . If the number of cycles of length  $i$  is denoted by  $c_i$  and the longest cycle has length  $k$ , then the cardinality of the conjugacy class of  $\pi$  in  $S_n$  is given by*

$$|C(\pi)| = n! \left( \prod_{i=1}^k i^{c_i} \prod_{i=1}^k c_i! \right)^{-1}.$$

**Proof.** The claim is elementary and follows by combinatorics as explained in [18], Section 2.3.1.  $\square$

**2.10 Lemma** *The length function  $\ell_c$  is asymptotically bounded by  $\ell_H$  in  $\mathcal{S}$ .*

**Proof.** We consider a non-trivial permutation  $\pi \in S_n$  which has  $m$  fixed points. Again we denote the number of cycles of length  $i$  of  $\pi$  by  $c_i$ . Then by assumption  $c_1 = m$ . By Stirling's formula for large  $n$  the estimate

$$\frac{1}{2}n \log n \leq \log n! \leq 2n \log n$$

holds.

Using Lemma 2.9 we obtain the trivial inequality

$$|C(\pi)| \leq n!(m!)^{-1}.$$

Therefore

$$\ell_c(\pi) \leq \frac{\sum_{i=1}^n \log i - \sum_{i=1}^m \log i}{\frac{1}{2}n \log n} \leq 2 \frac{\sum_{i=m+1}^n \log n}{n \log n} = 2 \frac{n-m}{n} = 2\ell_H(\pi). \quad \square$$

The converse inequality requires more effort.

**2.11 Lemma** *The size of a conjugacy class  $C(\pi)$  in  $S_n$  can be estimated as*

$$|C(\pi)| \geq \frac{n!}{m! \cdot 3^{\frac{1}{2}(n-m)} \gamma(\frac{1}{2}(n-m))},$$

where  $m$  is the number of trivial cycles in  $\pi$ .

**Proof.** We use the notation of Lemma 2.9, thus writing  $c_1 = m$ . First note that for  $k \geq 2$  obviously  $c_k! \leq \gamma(\frac{1}{2}kc_k)$ . Furthermore  $k^{c_k} \leq 3^{\frac{1}{2}kc_k}$ . This is true for  $c_k = 1$ , where we have  $k \leq k^{\frac{1}{2}k}$ . Because  $k^{c_k-1} \leq 3^{\frac{1}{2}k(c_k-1)}$  implies

$$k^{c_k} \leq 3^{\frac{1}{2}k(c_k-1)}k \leq 3^{\frac{1}{2}k(c_k-1)}3^{\frac{1}{2}k} = 3^{\frac{1}{2}kc_k},$$

induction on  $c_k$  works.

Now let  $P := \prod_{i=2}^k i^{c_i} \prod_{i=2}^k c_i!$ . We show for  $k = 2$  and afterwards for all remaining  $k$  by means of induction that  $P \leq 3^{\frac{1}{2}(n-m)} \gamma(\frac{1}{2}(n-m))$ . Obviously  $P = 2^{\frac{1}{2}(n-m)} (\frac{1}{2}(n-m))!$  for  $k = 2$ . If  $k \geq 3$  and the claim is true for  $k-1$ , then by the above preparation

$$\begin{aligned} P &\leq 3^{\frac{1}{2}(n-m-kc_k)} \gamma(\frac{1}{2}(n-m-kc_k)) k^{c_k} c_k! \\ &\leq 3^{\frac{1}{2}(n-m-kc_k)} \gamma(\frac{1}{2}(n-m-kc_k)) 3^{\frac{1}{2}kc_k} \gamma(\frac{1}{2}kc_k) \\ &\leq 3^{\frac{1}{2}(n-m)} \gamma(\frac{1}{2}(n-m)). \end{aligned}$$

Putting the parts together finishes the proof.  $\square$

**2.12 Lemma** *The size of a conjugacy class  $C(\pi)$  in  $S_n$  can be estimated as*

$$|C(\pi)| \geq \frac{n!}{m! \cdot \gamma(\frac{3}{4}(n-m))},$$

where  $m$  is the number of trivial cycles in  $\pi$  and  $n-m \geq 130$ .

*Proof.* We use Lemma 2.11 and the fact that

$$3^{\frac{1}{2}(n-m)} \gamma(\frac{1}{2}(n-m)) \leq \gamma(\frac{3}{4}(n-m)),$$

whenever  $n-m \geq 130$ .  $\square$

**2.13 Lemma** *The size of a conjugacy class  $C(\pi)$  in  $S_n$  can be estimated as*

$$|C(\pi)| \geq \frac{n!}{m! \cdot (n-m)!} = \binom{n}{m},$$

where  $m$  is the number of trivial cycles.

*Proof.* This is clear since each element in  $C(\pi)$  determines an  $m$ -element subset of  $[n]$  consisting of fixed points and each  $m$ -element subset arises this way.  $\square$

**2.14 Lemma** *In  $\mathcal{S}$  the length function  $\ell_H$  is locally asymptotically bounded by  $\ell_c$  in radius  $\frac{1}{4}$ .*

*Proof.* We use the same notation as in Lemma 2.10. To fit our radius requirements we further assume that  $m \geq \frac{3}{4}n$ . Let for the moment  $n-m \geq 130$  and  $k$  be the unique natural number such that

$$\frac{(k-1)n}{k} \leq m < \frac{kn}{k+1}.$$

This implies

$$\frac{n}{k+1} < n-m \leq \frac{n}{k},$$

and by our particular choice of  $m$  also  $k+1 > \frac{n}{n-m} \geq 4$ , whence  $k \geq 4$ . By the submultiplicativity of  $\gamma$

$$\gamma\left(\frac{kn}{k+1}\right) \cdot \gamma\left(\frac{3n}{4k}\right) \leq \gamma\left(\frac{kn}{k+1} + \frac{\frac{3(k+1)}{4k}n}{k+1}\right) \leq \gamma\left(\frac{(k + \frac{15}{16})n}{k+1}\right),$$



because  $\frac{3(k+1)}{4k} \leq \frac{15}{16}$  for  $k \geq 4$ .

Hence, by our choice of  $m$  and Lemma 2.12,

$$|C(\pi)| \geq \frac{n!}{m! \cdot \gamma(\frac{3}{4}(n-m))} \geq \frac{n!}{\gamma(\frac{kn}{k+1}) \cdot \gamma(\frac{3n}{4k})} \geq \frac{n!}{\gamma\left(\frac{(k+\frac{15}{16})n}{k+1}\right)}.$$

Now we shall use two more simple inequalities, namely

$$n - m < 2 \frac{n}{k+1}$$

and

$$\log n \leq 2 \log m.$$

Because  $n - m$  is not less than 130, in particular  $n - m \geq 64$ . Hence

$$n - \frac{(k + \frac{15}{16})n}{k+1} = \frac{n}{16(k+1)} > \frac{1}{32}(n-m) \geq 2,$$

and if  $l$  is the smallest integer larger than or equal to  $\frac{(k+\frac{15}{16})n}{k+1}$ , then  $l \leq n-1$ . Now the relations  $n-l \geq \frac{n}{16(k+1)} - 1$  and  $l+1 \geq \frac{kn}{k+1}$  follow easily. Because  $l \leq n-1$ , summation starting at  $l+1$  and ranging to  $n$  is not empty and, using all the small parts deduced so far, we obtain

$$\begin{aligned} \log |C(\pi)| &\geq \log(n!) - \log(l!) \\ &\geq \sum_{i=l+1}^n \log i \\ &\geq \left( \frac{n}{16(k+1)} - 1 \right) \log \left( \frac{kn}{k+1} \right) \\ &\geq \frac{1}{64}(n-m) \log m \\ &\geq \frac{1}{128}(n-m) \log n. \end{aligned}$$

Dividing by  $\log(n!) \leq 2n \log n$  yields

$$\ell_c(\pi) \geq \frac{n-m}{256n} = \frac{1}{256} \ell_H(\pi).$$

It remains to settle the case  $n - m \leq 129$ . Here we have

$$\log |C(\pi)| \geq \log(n!) - \log(m!) - \log(129!)$$

by Lemma 2.13. Because  $\log(129!) \leq 218$ ,

$$\begin{aligned} \ell_c(\pi) &\geq \frac{\log(n!) - \log(m!) - 218}{\log(n!)} \\ &\geq \frac{(n-m) \log m - 218}{2n \log n} \\ &\geq \frac{n-m}{4n} - \frac{109}{n \log n} \\ &\geq \frac{1}{5} \ell_H(\pi) \end{aligned}$$

for uncomfortably large  $n$ . □

**2.15 Lemma** *In  $\mathcal{S}$  the length function  $\ell_H$  is asymptotically bounded by  $\ell_c$ .*

*Proof.* By virtue of Lemma 2.14, it is enough to treat elements  $\pi$  of conjugacy length  $\ell_c(\pi) > \frac{1}{4}$ . By Stirling's formula for large  $n$  the estimate

$$\frac{99}{100}n \log n \leq \log n! \leq \frac{101}{100}n \log n$$

holds. Of course this statement remains true if we exchange the factorial for  $\gamma$ . Assume for the moment that  $m$  is not too small, so that the above estimate holds for  $m$  as well. We calculate, using Lemma 2.12,

$$\begin{aligned} \ell_c(\pi) &\geq \frac{\frac{99}{100}n \log n - \frac{101}{100}m \log m - \frac{101}{100} \cdot \frac{3}{4}(n-m) \log(\frac{3}{4}(n-m))}{\frac{101}{100}n \log n} \\ &\geq \frac{99n \log n - 101m \log n - \frac{303}{4}(n-m) \log n}{101n \log n} \\ &= \frac{\frac{93}{4}n - \frac{101}{4}m}{101n} \\ &= \frac{93n - 32m - 69m}{404n} \\ &> \frac{69n - 69m}{404n} \\ &= \frac{69}{404}\ell_H(\pi), \end{aligned}$$

where we used  $m < \frac{3}{4}n$ . The proportion of elements remaining is small in the sense that for these  $m$  is less than some constant  $K$ , independent of  $n$ . Now the argument used at the end of the proof of Lemma 2.14 applies.  $\square$

**2.16 Theorem** *In  $\mathcal{S}$  and  $\mathcal{A}$  alike the length functions  $\ell_H$  and  $\ell_c$  are asymptotically equivalent.*

*Proof.* The conjugacy classes of  $S_n$  behave in two different ways. Either they correspond to exactly one conjugacy class in  $A_n$ , or they split into two classes in  $A_n$ . In the first case the size of the conjugacy class stays the same, whereas in the second case it splits into two parts of equal size. (Confer [18], Paragraph 2.3.2.) Now Lemma 2.10 and Lemma 2.15 apply.  $\square$

## 2.3 Length functions on linear groups

Given a (finite dimensional) vector space  $V$  we write  $\text{GL}(V)$  for all bijective linear transformations of  $V$ ,  $\text{SL}(V)$  for all linear transformations of  $V$  of determinant 1. When  $V = F^n$  for some field  $F$  we use notation  $\text{GL}_n(F)$  and the like, which reduces further to  $\text{GL}_n(q)$  etc. when  $F$  is the finite field  $\mathbb{F}_q$  of order  $q$ .

We shall deal in particular with linear groups over finite fields and introduce the symbols  $\mathcal{GL}(q)$  for the class of all general linear groups defined over the field  $\mathbb{F}_q$  and  $\mathcal{GL}$  for the union of these, where  $q$  ranges over all prime powers. Exchanging general for special yields  $\mathcal{SL}(q)$  and  $\mathcal{SL}$ . If  $V$  is a vector space over a field  $F$  we will write 1 for the identity mapping  $V \rightarrow V$  and write simply  $\alpha$  for the mapping  $\alpha \cdot 1$ , where  $\alpha \in F$ .

**2.17 Proposition** *Let  $V$  be a vector space of dimension  $n$ . Then*

$$\ell_r(g) := n^{-1} \operatorname{rk}(1 - g)$$

*is an invariant length function on  $\operatorname{GL}(V)$ .*

**Proof.** It is clear that  $\ell_r$  takes its values in  $[0, 1]$  and that  $\ell_r(g) = 0$  if and only if  $g = 1$ . Moreover, if  $g, h$  are in  $\operatorname{GL}(V)$ , then

$$\operatorname{rk}(\operatorname{id} - g) = \operatorname{rk}(-g^{-1}(1 - g)) = \operatorname{rk}(1 - g^{-1})$$

and

$$\begin{aligned} \operatorname{rk}(1 - gh) &= \operatorname{rk}((h^{-1} - g)h) \\ &= \operatorname{rk}((1 - g) - (1 - h^{-1})) \\ &\leq \operatorname{rk}(1 - g) + \operatorname{rk}(1 - h^{-1}) \\ &= \operatorname{rk}(1 - g) + \operatorname{rk}(1 - h). \end{aligned}$$

The invariance of  $\ell_r$  follows from

$$\operatorname{rk}(1 - hgh^{-1}) = \operatorname{rk}(h(1 - g)h^{-1}) = \operatorname{rk}(1 - g). \quad \square$$

We call the function  $\ell_r$  the **rank length**.

Now the following conclusion is rather obvious.

**Proof of Proposition 2.6.** The symmetric group embeds as the subgroup of permutation matrices into  $\operatorname{GL}(V)$ . If  $\pi$  consists of the cycles  $\pi_1, \dots, \pi_l$  then the corresponding permutation matrix  $P_\pi$  equals the direct sum  $P_{\pi_1} \oplus \dots \oplus P_{\pi_l}$ . The eigenvalues of a permutation matrix are all 1 and  $\operatorname{rk}(\operatorname{id} - \pi_i) = k - 1$  if  $\pi_i$  has length  $k$ . Hence  $\ell_r$  is the restriction of the rank length to permutations.  $\square$

We want to prove a similar result for general linear groups over finite fields as we obtained for permutation groups in the last subsection. As it turns out it is necessary to gain some independence of the base field. We therefore introduce the **Jordan length** (the name of which is explained below.)

$$\ell_J := (\ell_{\operatorname{rGL}(V)/Z(\operatorname{GL}(V))})^{\operatorname{GL}(V)}.$$

A more explicit description of  $\ell_J$  is

$$\ell_J(g) = n^{-1} \cdot \inf_{\alpha \in F^\times} \operatorname{rk}(\alpha - g),$$

as the center of  $\operatorname{GL}(V)$  is isomorphic to  $F^\times$ .

From now on we shall write

$$m_g := \sup_{\alpha \in F^\times} \dim(\ker(\alpha - g)),$$

whenever  $g$  is an element in a linear group over a field  $F$ . With this definition yet another characterization of the Jordan length is

$$\ell_J(g) = \frac{n - m_g}{n}.$$

**2.18 Proposition** *Let  $g$  be an element in  $\mathrm{GL}(V)$ . If  $\ell_r(g) \leq \delta$ , then  $\ell_J(g) \geq \min\{(1 - \delta), \delta\}$ .*

*Proof.* Let  $m = \mathrm{rk}(1 - g)$ . In the easiest case  $\ell_J(g) = \ell_r(g) \geq \delta$ . Hence we can assume  $m \neq m_g$ . Then of course  $m + m_g \leq n$  and

$$\ell_J(g) = \frac{n - m_g}{n} \geq \frac{n - (n - m)}{n} = \frac{m}{n} = 1 - \ell_r(g) \geq 1 - \delta$$

follows.  $\square$

**2.19 Corollary** *Let  $g$  be an element in  $\mathrm{GL}(V)$ . If  $\ell_r(g) \leq \frac{1}{2}$ , then  $\ell_r(g) = \ell_J(g)$ .*

*Proof.* By definition  $\ell_J(g) \leq \ell_r(g)$ .  $\square$

We cite almost verbatim from the introduction of the Jordan decomposition on pp. 395, 396 in [11]. Each  $g \in \mathrm{SL}_n(q)$  equals the commuting product  $su$  of a semisimple element  $s$  and a unipotent element  $u$ , this being called the Jordan decomposition.

We denote by  $J_k$  the unipotent  $k \times k$  Jordan matrix

$$\begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$$

The matrix  $(J_i^{m_{ij}})$  is defined as the matrix consisting of Jordan blocks  $J_i$  which appear  $m_{ij}$  times. Then  $g$  is conjugate to a matrix

$$\alpha_1(J_i^{m_{i1}}) \oplus \dots \oplus \alpha_r(J_i^{m_{ir}}) \oplus \lambda_1 \otimes (J_i^{n_{i1}}) \oplus \dots \oplus \lambda_t \otimes (J_i^{n_{it}}),$$

where  $\alpha_j \in \mathbb{F}_q^\times$  and the  $\lambda_j$  are irreducible matrices. In this representation we can assume that  $m := m_{11} \geq m_{21} \geq \dots \geq m_{1r}$ . That is  $m$  counts the maximal number of Jordan blocks of size 1 to an eigenvalue  $\alpha_1 \in \mathbb{F}_q$  of  $g$ .

Fortunately  $m$  and  $m_g$  compare well enough to relate the results in [11] to the Jordan length. A first application can be seen in the following theorem.

**2.20 Theorem** *The pseudo length functions  $\ell_c$  and  $\ell_J$  are asymptotically equivalent in  $\mathcal{GL}$  and  $\mathcal{SL}$ .*

*Proof.* We first consider the case of special linear groups. Because  $m$  counts the maximal number of Jordan blocks of size 1 to a fixed eigenvalue of  $g$ ,  $m_g$  is maximal when we can find the maximal number of Jordan blocks of minimal size in the Jordan decomposition of  $g$ . The minimal size remaining for our choice is 2 and hence  $m_g - m \leq \frac{n-m}{2}$ . Now we can deduce like in the proof of Proposition 2.7 that

$$\frac{n - m_g}{n} \geq \frac{n - m}{2n} \geq c' \frac{\log |C(g)|}{|\mathrm{SL}_n(q)|}$$

for a constant  $c'$ , using also Lemma 5.3 in [11]. Thus  $\ell_c$  is asymptotically bounded by  $\ell_J$ .

Lemma 5.4, *ibid.*, states that there is a universal constant  $c$  such that whenever  $1 \neq g \in \mathrm{SL}_n(q)$  and  $k \geq \frac{cn}{n-m}$ , then  $C(g)^k = \mathrm{SL}_n(q)$ . We can assume  $c$  an integer and  $k$  minimal such that  $k = \varepsilon c \frac{n}{n-m_g} \geq \varepsilon c \frac{n}{n-m}$ , where the error  $\varepsilon$  is definitely less than 2. Now  $|\mathrm{SL}_n(q)| \leq |C(g)|^k$  and

$$k\ell_c(g) = \frac{\log(|C(g)|^k)}{\log|\mathrm{SL}_n(q)|} \geq \frac{\log|C(g)^k|}{\log|\mathrm{SL}_n(q)|} = 1.$$

This implies

$$\ell_c(g) \geq k^{-1} = \varepsilon^{-1} c^{-1} \frac{n-m_g}{n} \geq \frac{1}{2} c^{-1} \ell_J(g).$$

By comparison of the sizes of conjugacy classes in  $\mathrm{SL}_n(q)$  and  $\mathrm{GL}_n(q)$  the claim follows for  $\mathcal{GL}(q)$ . Because the above argument is independent of  $q$ , we have proved the theorem for  $\mathcal{SL}$  and  $\mathcal{GL}$ .  $\square$

We remark that the proof of the preceding theorem, which is based on deep results of Liebeck-Shalev, can be done by elementary methods, estimating the size of centralizers of matrices with respect to the number and sizes of Jordan blocks involved. The proof then develops similarly to the case of symmetric groups, apart from the point that the complexity of the values in the estimates is  $n^2$  instead of  $n \log n$ , which turns out to be easier to handle. The elementary argument requires several pages, though, and therefore is omitted.

### 3 Ultraproducts of finite simple groups

We follow the notation in the first chapter of [5] when addressing finite simple groups of Lie type. That is given a vector space  $V$  with a bilinear or Hermitian form we write  $\mathrm{GI}(V)$  for all isometries of  $V$  and  $\mathrm{SI}(V) := \mathrm{GI}(V) \cap \mathrm{SL}(V)$  (with the exception of orthogonal forms in characteristic 2, which is explained below.) Thus for the trivial bilinear form  $\mathrm{GL}(V) = \mathrm{GI}(V)$ . We write  $\mathrm{Sp}(V) := \mathrm{SI}(V) = \mathrm{GI}(V)$  for a symplectic bilinear form on  $V$ ,  $\mathrm{GU}(V) := \mathrm{GI}(V)$  and  $\mathrm{SU}(V) := \mathrm{SI}(V)$  for a Hermitian form on  $V$  and  $\mathrm{GO}(V) := \mathrm{SI}(V)$  and  $\mathrm{SO}(V) := \mathrm{SI}(V)$  for a symmetric bilinear form on  $V$  in odd characteristic. (Over characteristic 2 the group  $\mathrm{SO}(V)$  is defined as the kernel of the Dickson invariant.) Furthermore  $\Omega(V) := \mathrm{GO}(V)' = \mathrm{SO}(V)'$ , the commutator subgroup of  $\mathrm{SO}(V)$ . We prefix one more letter to denote the quotients of all these groups by their center, thus writing  $\mathrm{PSL}(V)$ ,  $\mathrm{PSp}(V)$  and so on.

When dealing with ultrafilters we introduce the following abbreviating notation. Let  $\mathbf{u}$  be an ultrafilter on a set  $I$ . We say that a property  $P$  holds  **$\mathbf{u}$ -almost everywhere** or for  **$\mathbf{u}$ -almost all**  $i$  if the set  $\{i \in I \mid P(i)\}$  is in  $\mathbf{u}$ . We also write  $P(i)$   $[\mathbf{u}]$  in this situation.

If  $A_i$  is a family of algebraic structures indexed by  $I$ , we write  $\prod_{i \rightarrow \mathbf{u}} A_i$  for the ultraproduct or, when the right index is understood, only  $\prod_{\mathbf{u}} A_i$ . A similar notation is used for limits along an ultrafilter, namely  $\lim_{i \rightarrow \mathbf{u}} a_i$ , or  $\lim_{\mathbf{u}} a_i$  to save symbols.

In the following we fix a non-principal ultrafilter  $\mathbf{u}$  on the natural numbers.

If  $G_n$  are groups equipped with a generic (pseudo) length function  $\ell$  we write  $\mathbf{G}$  for the ultraproduct  $\prod_{\mathbf{u}} G_n$  and  $\mathbf{g}$  for an element represented by a sequence  $(g_n)_{n \in \mathbb{N}}$ . Moreover we let

$$\ell(\mathbf{g}) := \lim_{\mathbf{u}} \ell(g_n).$$

Then  $\mathbf{N} := \{\mathbf{g} \in \mathbf{G} \mid \ell(\mathbf{g}) = 0\}$  is a normal subgroup, as can be deduced from the properties of pseudo length functions.

**3.1 Proposition** *Let  $\mathcal{G} = \{G_n \mid n \in \mathbb{N}\}$  be a collection of finite non-abelian simple groups. Then the group  $\mathbf{G}/\mathbf{N}$  is simple.*

*Proof.* We show that if  $\ell_c(\mathbf{g}) = \varepsilon > 0$  for  $\mathbf{g} \in \mathbf{G}$ , then already  $\mathbf{N}(\mathbf{g}) = \mathbf{G}$ . By Theorem 1.1 in [11] there is a universal constant  $c$  such that whenever  $G$  is a finite non-abelian simple group and  $1 \neq g \in G$ , then  $C(g)^m = G$  for any  $m \geq c \frac{\log |G|}{\log |C(g)|}$ . By our assumption  $\frac{\log |G_{\omega(i)}|}{\log |C(g_i)|} \leq K [\mathbf{u}]$ . Hence for  $m \geq cK$ ,  $C(g_i)^m = G_{\omega(i)} [\mathbf{u}]$  or equivalently  $C(\mathbf{g})^m = \mathbf{G}$ . We conclude that the set of all elements of zero length in  $\mathbf{G}$  is a maximal normal subgroup and thus  $\mathbf{G}$  divided by this subgroup is simple.  $\square$

In fact the converse is also true. If a quotient of a direct product of finite simple non-abelian groups is simple, then it is a quotient as in the preceding theorem for some choice of ultrafilter. Confer [12], proof of Proposition 3, for the argument.

Instead of considering ultraproducts of projective special linear groups we can look at ultraproducts of arbitrary finite simple groups of Lie type. Note that in the theory of these groups up to a certain rank exceptional cases occur, which may require special treatment. If a statement in this section is wrong, it becomes true if we restrict ourselves to groups of large enough rank.

### 3.1 Some geometry

We need some basic geometric lemmas to prepare what follows. We use the symbol  $\oplus$  to denote the orthogonal direct sum. The next lemma is essentially Corollary 2.3 in [9].

**3.2 Lemma** *Let  $V$  be a finite dimensional vector space with non-degenerate bilinear or Hermitian form  $(\cdot, \cdot)$  and  $W$  some subspace. If  $\varphi \in W^*$ , then there is  $v \in V$  such that for all  $w \in W$  the equation  $(w, v) = \varphi(w)$  holds.*

*Proof.* Because  $V$  is non-degenerate,  $0 = \text{rad}(V) = \ker(v \mapsto (w \mapsto (w, v)))$ . Hence for all  $\varphi \in V^*$  there is  $v \in V$  such that  $\varphi(w) = (w, v)$  for all  $w \in V$ . Let now  $W$  be a subspace with basis  $w_1, \dots, w_k$ , which extends to a basis  $w_1, \dots, w_n$  of  $V$ . Extend  $\varphi \in W^*$  to a linear form  $\tilde{\varphi}$  on  $V$  by  $\tilde{\varphi}(w_i) = 0$ ,  $i > k$  and find  $v$  such that  $\varphi(w) = (w, v)$  for all  $w \in W$ .  $\square$

**3.3 Lemma** *Let  $V$  be a finite dimensional vector space with non-degenerate bilinear or Hermitian form  $(\cdot, \cdot)$  and  $W$  a subspace. Let  $R$  be the radical of  $W$  and  $W'$  a complement of  $R$  in  $W$ . Then there is a subspace  $W''$  of  $V$ , which satisfies  $\dim(W'') = \dim(R)$  and  $(W'' \oplus W^\perp) \oplus W'$ . In particular  $W'$  and  $U := W'' \oplus W^\perp$  are non-degenerate.*

**Proof.** We use Lemma 3.2. Let  $w_1, \dots, w_r$  be a basis of  $R$  and  $w_1, \dots, w_k$  an extension to a basis of  $W + W^\perp$ . For  $r = 0$  there is nothing to show since  $W \oplus W^\perp = V$ . Now assume  $r \geq 1$  and define  $\varphi_1 \in (W + W^\perp)^*$  by  $\varphi_1(w_1) = 1$  and  $\varphi_1(w_i) = 0$  otherwise. Then there is  $v_1 \in V$  such that  $(w_1, v_1) = 1$  and  $v_1 \perp \langle w_2, \dots, w_k \rangle$ . Now  $\dim(\text{rad}(W + W^\perp + \langle v_1 \rangle)) = r - 1$  and we can proceed inductively defining  $\varphi_l \in (W + W^\perp + \langle v_1, \dots, v_{l-1} \rangle)^*$  by  $\varphi_l(w_l) = 1$  and  $\varphi_l(w_i) = 0$ ,  $\varphi_l(v_i) = 0$  for the remaining basis vectors. In the end this gives us  $v_1, \dots, v_r$  such that  $W'' := \langle v_1, \dots, v_r \rangle$  meets our expectations.  $\square$

**3.4 Lemma** *Let  $V$  be a finite dimensional vector space over a field  $F$  with bilinear or Hermitian form  $B = (\cdot, \cdot)$ . We exclude the case that  $\text{char}(F) = 2$  and  $B$  symmetric. Let  $U$  be a non-degenerate subspace of  $V$ . Then the subgroup  $H := \{g \in \text{SI}(V) \mid g|U^\perp = \text{id}_{U^\perp}\}$  of  $\text{SI}(V)$  is isomorphic to  $\text{SI}(U)$ .*

**Proof.** Let  $g$  be in  $\text{SI}(U)$ . We define  $\varphi(g) := g \oplus \text{id}_{U^\perp}$ . Obviously  $\varphi(g) \in \text{GL}(V)$  and  $\varphi$  is a homomorphism. If  $v, w$  are in  $V$  and decompose as  $v = v_U + v_{U^\perp}$ ,  $w = w_U + w_{U^\perp}$  with respect to the direct sum  $V = U \oplus U^\perp$ , then

$$\begin{aligned} (\varphi(g)(v), \varphi(g)(w)) &= (g(v_U), g(w_U)) + (v_{U^\perp}, w_{U^\perp}) \\ &= (v_U, w_U) + (v_{U^\perp}, w_{U^\perp}) \\ &= (v, w), \end{aligned}$$

whence  $\varphi(g) \in \text{GI}(V)$ . If  $\det(g) = 1$ , obviously also  $\det(\varphi(g)) = 1$  and  $\varphi(g) \in \text{SI}(V)$ . Because elements in  $H$  stabilize  $U^\perp$ ,  $g|U$  is guaranteed to stabilize  $U$ . Now the inverse mapping of  $\varphi$  is easily verified to be  $g \mapsto g|U$ .  $\square$

**3.5 Lemma** *Let  $V$  be a finite dimensional vector space over a field  $F$  of odd characteristic with non-degenerate symmetric bilinear form  $(\cdot, \cdot)$ . Let  $U$  be a non-degenerate subspace. Then the subgroup  $H := \{g \in \Omega(V) \mid g|U^\perp = \text{id}_{U^\perp}\}$  of  $\Omega(V)$  is isomorphic to  $\Omega(U)$ .*

**Proof.** Let  $g \in H$ . Because  $g|U \in \text{GO}(U)$ , it can be written as a product of reflections  $s_{u_1}, \dots, s_{u_k}$  in  $\text{GO}(U)$ , where the reflection is along the hyperplane  $\langle u_i \rangle^\perp$  (and  $u_i$  non-degenerate). In particular  $u_i \in U$  for all  $i$ . Each reflection  $s_{u_i}$  is given explicitly by the expression

$$s_{u_i}(w) = w - Q(u_i)^{-1}(w, u_i)u_i,$$

where  $Q$  is the associated quadratic form. From orthogonality we deduce that  $s_{u_i}^V := s_{u_i} \oplus \text{id}_{U^\perp}$  is a reflection in  $\text{GO}(V)$ .

By Lemma 3.4 we know that  $g|U \in \text{SO}(U)$ . By [9], Theorem 9.7

$$\Omega(V) = \text{SO}(V) \cap \ker(\vartheta),$$

where  $\vartheta$  is the spinor norm. (Confer [9], Chapter 9, pp. 75, 76.) We see

$$1 = \vartheta(g) = \vartheta(s_{u_1}^V \dots s_{u_k}^V) = Q(u_1) \dots Q(u_k) F^{\times 2} = \vartheta(s_{u_1} \dots s_{u_k}) = \vartheta(g|U)$$

and conclude  $g|U \in \Omega(U)$ . The claim follows.  $\square$

**3.6 Lemma** *Let  $V$  be a vector space of dimension  $n$  over a perfect (e. g. finite) field of characteristic 2. Let  $Q$  be a regular (i. e.  $\dim(\text{rad}(V)) \in \{0, 1\}$ ) quadratic form on  $V$  and  $B = (\cdot, \cdot)$  the associated bilinear form. Let  $U$  be a regular subspace of  $V$  and  $H := \{g \in \Omega(V) \mid g|U^\perp = \text{id}_{U^\perp}\}$ . Then  $H$  is isomorphic to  $\Omega(U)$ .*

**Proof.** We note that  $H$  is isomorphic to a subgroup of  $\mathrm{GO}(U)$  by  $g \mapsto g|U$ . We divide the proof according to the dimension of the radical of  $V$  and assume first that  $V$  is defective. Then by Theorem 14.2 in [9]  $\mathrm{GO}(V)$  is isomorphic to  $\mathrm{Sp}(V_1)$  for a complement  $V_1$  of  $\mathrm{rad}(V)$  and the action of  $\mathrm{GO}(V)$  on  $\mathrm{rad}(V)$  is trivial. We see that whether  $U$  is defective or not, the proof of Lemma 3.4 applies.

Now assume that  $V$  is non-defective. By [9], Proposition 14.23

$$\Omega(V) = \mathrm{SO}(V) \cap \ker(\vartheta),$$

where  $\vartheta$  is the spinor norm. Note that  $\mathrm{SO}(V)$  is the kernel of the Dickson pseudoinvariant  $\delta : \mathrm{GO}(V) \rightarrow \mathbb{F}_2$ , or equivalently the subgroup of all products of an even number of orthogonal transvections. Now if  $g \in H$ , then  $g|U \in \mathrm{GO}(U)$  and hence is a product of transvections  $t_{u_1}, \dots, t_{u_k}$ , see [9], Theorem 14.16. Each orthogonal transvection is described explicitly by

$$t_{u_i}(w) = w + Q(u_i)^{-1}(w, u_i)u_i.$$

We implicitly used that none of the vectors  $u_i$  is singular. By extending  $g|U$  to the whole of  $V$  as in Lemma 3.5,  $k$  is necessarily even. The proof now continues as in Lemma 3.5.  $\square$

## 3.2 Normal subgroups of ultraproducts of finite simple groups

In [7] the following result was proved.

**3.7 Theorem ([7], Theorem 1.1)** *Let  $\mathfrak{u}$  be a non-principal ultrafilter on the natural numbers. Then the set of normal subgroups of  $\prod_{\mathfrak{u}} A_n$  is linearly ordered.*

Another formulation of this statement can be found in [1] as Theorem 3.

Note at this point, as we start to develop the generalization of Theorem 3.7, that even if we treat different cases separately, the statements can be read as for arbitrary finite simple groups. This follows simply from the fact that we have finitely many possible choices, namely permutation groups and the different types of matrix groups. The chosen ultrafilter used to build an ultraproduct  $\mathbf{G}$  of finite simple groups then “decides” that  $\mathbf{G}$  is isomorphic to an ultraproduct of groups in exactly one of these families.

We obtain the following proposition for groups of Lie type almost instantly.

**3.8 Proposition** *Let  $G_n$  be finite simple groups of Lie type for all  $n \in \mathbb{N}$ . Let  $\mathfrak{u}$  be a non-principal ultrafilter on the natural numbers. If  $\mathbf{G} = \prod_{\mathfrak{u}} G_n$  and the rank of the groups  $G_n$  is bounded, then  $\mathbf{G}$  is simple.*

**Proof.** Suppose that the rank of the groups in question is bounded by  $N$ . Let  $1 \neq g \in G_n$ . Using the constant  $c$  of Theorem 1.1 in [11] we see that for

$$m \geq c \log |G_n| \log |C(g)|^{-1}$$



already  $C(g)^m = G_n$ . When  $G_n$  is a group over the field  $\mathbb{F}_q$ , its order is at most  $q^{c'n^2}$  for a universal constant  $c'$ . On the other hand a non-trivial conjugacy class in  $G_n$  has at least  $q$  elements. Hence it suffices for  $m$  to be larger than  $c'N^2$  to ensure  $C(g)^m = G_n$  for any  $n$ .

If we choose  $1 \neq \mathbf{g} \in \mathbf{G}$  arbitrarily, then  $C(g_n)^m = G_n$  for  $\mathbf{u}$ -almost all  $n$ . Hence  $C(\mathbf{g})^m = \mathbf{G}$  and consequently  $N(\mathbf{g}) = \mathbf{G}$ . Therefore  $\mathbf{G}$  contains no proper normal subgroups, whence it is simple.  $\square$

We take Theorems 3.7 and Proposition 3.8 as a motivation to prove the following more general Theorem 3.9. In the proof we follow a similar route as the authors of [7] in the proof of their Theorem 1.1.

**3.9 Theorem** *Let  $G_n$  be finite simple groups of Lie type for all  $n \in \mathbb{N}$ . If  $\mathbf{G} := \prod_{\mathbf{u}} G_n$ , then the set  $\mathfrak{N}$  of normal subgroups of  $\mathbf{G}$  is totally ordered.*

In view of Theorem 3.8 we only need to take care of classical groups of unbounded rank.

First consider the general situation that we are given an ultraproduct  $\mathbf{G} = \prod_{\mathbf{u}} G_i$  of arbitrary groups  $G_i$  with length function  $\ell_i$ . We define an ordering of the non-trivial elements of  $\mathbf{G}$  by  $\mathbf{g} \preceq \mathbf{h}$  if

$$\lim_{\mathbf{u}} \frac{\ell_i(g_i)}{\ell_i(h_i)} < \infty.$$

**3.10 Lemma** *Let  $\mathbf{g}$  and  $\mathbf{h}$  be non-identity elements of the ultraproduct  $\mathbf{G}$  of groups  $G_i$ . Then  $\mathbf{g} \in N(\mathbf{h})$  implies  $\mathbf{g} \preceq \mathbf{h}$ .*

**Proof.** If  $\mathbf{g} \in N(\mathbf{h})$  there is some integer  $k$  such that  $\mathbf{g}$  is a product of  $k$  conjugates of  $\mathbf{h}^{\pm 1}$ . Therefore  $g_i$  is a product of  $k$  conjugates of  $h_i^{\pm 1}$  for  $\mathbf{u}$ -almost all  $i$ . By the properties of invariant length functions

$$\ell_i(g_i) \leq k\ell_i(h_i^{\pm 1}) = k\ell_i(h_i) \quad [\mathbf{u}].$$

Hence

$$\lim_{\mathbf{u}} \frac{\ell_i(g_i)}{\ell_i(h_i)} \leq k$$

follows and we are done.  $\square$

We want to show that for finite simple groups of Lie type and the Jordan length the converse of the previous lemma is true. The following statement is a summary of results from [11].

**3.11 Lemma** *Let  $G$  be a quasisimple group of Lie type of rank  $n$  and  $1 \neq g \in G$ . There is a constant  $c$ , independent of  $G$  and  $g$  such that  $C(g)^m = G$  for all  $m \geq \frac{cn}{n-m_g}$ .*

**Proof.** For special linear groups use Lemma 5.4 in [11], for symplectic and orthogonal groups Lemma 6.4 and for unitary groups Section 7 *ibid*.  $\square$

**3.12 Lemma** *Let  $G$  be an ultraproduct of finite simple groups of Lie type equipped with the Jordan length. Then  $g \preceq h$  implies  $g \in N(h)$  for all non-trivial elements  $g, h \in G$ .*

*Proof.* By the hypothesis there is a natural number  $k$  such that  $\frac{n-m_{g_n}}{n-m_{h_n}} \leq k$  for  $\mathbf{u}$ -almost all  $n$ .

Let  $G = \text{SI}(V)$ , where  $V$  has dimension  $n$ . We can exclude the case when the characteristic of the field of definition is 2 and  $V$  is a defective quadratic space from the following considerations, since under that assumptions  $G = \text{GO}(V)$  is isomorphic to a symplectic group. Assume that  $\frac{n-m_g}{n-m_h} \leq k$  for some non-trivial elements  $g, h \in G$  such that  $n - m_g = \text{rk}(1 - g)$  and  $n - m_h = \text{rk}(1 - h)$ , that is their rank length and Jordan length are the same. We define  $W := \ker(1 - g) \cap \ker(1 - h)$ . If  $W'$  is a complement of  $\text{rad}(W)$  in  $W$ , following Lemma 3.3 there is subspace  $W''$  such that  $U := W'' \oplus W^\perp$  is non-degenerate and  $W' = U^\perp$ . Obviously  $g$  and  $h$  act as the identity on  $U^\perp$ . Then  $g|U$  and  $h|U$  are in  $H := \text{SI}(U)$ . We perform some calculations of dimensions to see

$$\dim(W^\perp) = n - \dim(W) \leq n - (m_g + m_h - n) = (n - m_g) + (n - m_h)$$

and

$$\dim(\text{rad}(W)) \leq n - \dim(W) \leq (n - m_g) + (n - m_h).$$

This together with the introductory remarks implies

$$\dim(U) = \dim(W^\perp) + \dim(\text{rad}(W)) \leq 2(n - m_g) + 2(n - m_h) \leq (2k + 2)(n - m_h).$$

Therefore the Jordan length of  $h|U$  estimates as

$$\ell_J(h|U) = \frac{\dim(U) - (\dim(U) - (n - m_h))}{\dim(U)} = \frac{n - m_h}{\dim(U)} \geq \frac{1}{2k + 2}.$$

By Lemma 3.11 there is a constant  $c$ , independent of the hypotheses, such that  $((h|U)^H)^m = H$  for  $m \geq c(2k + 2)$  and consequently  $g|U$  is a product of  $m$  conjugates of  $h|U$  inside  $H$ . Like in Lemma 3.4 we extend the elements occuring in this product to elements in  $G$ , thereby extending  $g|U$  to  $g$  and  $h|U$  to  $h$ . Thus the conclusion remains true in  $G$  and also when returning attention to the finite simple group  $G/Z(G)$ .

Because the prototype  $G/Z(G)$  was independent of  $n$  and the hypotheses did hold for almost all  $n$ ,  $g_n$  is a product of  $m \geq c(2k + 2)$  conjugates of  $h_n$  in  $G_n$  for almost all  $n$ . Hence

$$g \in C(h)^m \subset N(h),$$

which we had to prove.  $\square$

**3.13 Corollary** *If  $g$  and  $h$  are non-identity elements in  $G$ , the statements  $g \in N(h)$  and  $g \preceq h$  are equivalent.*

The last preparation we need is Lemma 2.2 in [7], which for the sake of completeness we cite with proof.

**3.14 Lemma** *Let  $G$  be any group. Then the set of normal subgroups of  $G$  is linearly ordered by inclusion if and only if the set of normal closures of non-identity elements in  $G$  is.*

**Proof.** The first implication is trivial. For the converse assume that  $N$  and  $M$  are normal subgroups of  $G$  such that  $N \not\subset M$ . Let  $g \in N \setminus M$  and observe that necessarily  $N(g) \not\subset N(h)$  for all  $h \in M$ . Thus  $N(h) \subset N(g)$  for all  $h \in M$ , and  $M \subset N$  follows.  $\square$

**Proof of Theorem 3.9.** We define a quasiorder on the set  $\mathbf{L} := \prod_{\mathbf{u}} [n]$  by  $\mathbf{a} \preceq \mathbf{b}$  if  $a_n \leq b_n$  for  $\mathbf{u}$ -almost all  $n$ . We let furthermore  $\mathbf{a} \equiv \mathbf{b}$ , whenever

$$0 < \lim_{\mathbf{u}} \frac{a_n}{b_n} < \infty.$$

Then  $\equiv$  is a convex equivalence relation and the quotient space  $\mathbf{L}/\equiv$  is totally ordered.

By the foregoing considerations, culminating in Corollary 3.13, the set of normal closures of elements in  $\mathbf{G}$  is order isomorphic to  $\mathbf{L}/\equiv$ . Lemma 3.14 shows that the set of normal subgroups of  $\mathbf{G}$  is linearly ordered by inclusion if and only if the set of normal closures of elements of  $\mathbf{G}$  is. Now Theorem 3.9 follows.  $\square$

After the main theorem explaining the ordering of normal subgroups in ultraproducts of finite simple groups is established, we take a closer look at those.

**3.15 Lemma** *Let  $\mathbf{G}$  be an ultraproduct of finite simple groups. A normal subgroup  $N \subset \mathbf{G}$  is of the form  $N(\mathbf{g})$  for some  $\mathbf{g} \in \mathbf{G} \setminus \{1\}$  if and only if  $N$  has a predecessor with respect to the ordering of normal subgroups.*

**Proof.** Using Theorem 3.9, it is easy to see that the set  $\{\mathbf{h} \in \mathbf{G} \mid \mathbf{h} \prec \mathbf{g}\}$  is a maximal normal subgroup of  $N(\mathbf{g})$ . Conversely, let  $N$  be a normal subgroup of  $\mathbf{G}$  with a predecessor  $N_0$ . Then there exists  $\mathbf{g} \in N \setminus N_0$ . Since  $N_0 \subsetneq N(\mathbf{g}) \subset N$ , we conclude  $N = N(\mathbf{g})$ , since  $N_0$  is the predecessor of  $N$ .  $\square$

If  $\mathbf{g} \in \mathbf{G}$  we denote the predecessor of  $N(\mathbf{g})$  by  $N_0(\mathbf{g})$ .

**3.16 Proposition** *Let  $\mathbf{G}$  be an ultraproduct of finite simple groups. Then every normal subgroup  $N$  in  $\mathbf{G}$  is perfect. In particular every element in  $N$  is itself a commutator of elements in  $N$ .*

**Proof.** For a start assume that  $\mathbf{G}$  is an ultraproduct of alternating groups. Given  $\mathbf{g}$  without loss of generality each  $g_n$  is an element in  $A_n$ . We can consider  $g_n$  as an element in the alternating group,  $A_k$  say, of the support of  $g_n$ . The famous paper [14] of Ore (which led to the Ore Conjecture) implies that  $g_n$  is a commutator of elements  $x_n$  and  $y_n$  in  $A_k$ , as long as  $k \geq 5$ , which we can assume without worry. Interpreting  $x_n$  and  $y_n$  as elements in  $A_n$  we automatically have  $\ell_H(x_n), \ell_H(y_n) \leq \ell_H(g_n)$ . Therefore  $\mathbf{x}, \mathbf{y} \preceq \mathbf{g}$ , which entails  $\mathbf{x}, \mathbf{y} \in N(\mathbf{g}) \subset N$ . By the same reasoning  $[\mathbf{x}, \mathbf{y}] \in N$ .

In the case of groups of Lie type we go the same way. Since Liebeck et al. solved the Ore conjecture in [10] also for quasisimple groups of Lie type, we can safely work in these groups. If  $g_n \in \mathrm{SL}_n(q)$  there is  $\alpha \in \mathbb{F}_q^\times$  such that  $\ker(\alpha^{-1} - g_n)$  has a higher dimension than  $\ker(\beta^{-1} - g_n)$  for any other  $\beta$ . We fix a complement of  $\ker(1 - \alpha g_n)$  and call this subspace  $U$ . It is clear that  $U$  is invariant under the action of  $g_n$  and  $g_n|_U \in \mathrm{SL}(U)$ . Hence we find  $x_n$  and  $y_n$  in  $\mathrm{SL}(U)$  such that  $g_n|_U = [x_n, y_n]$ . Using  $x_n$  and  $y_n$  only for building the

commutator we can simply assume  $\ell_J(x_n) = \ell_r(x_n)$  and  $\ell_J(y_n) = \ell_r(y_n)$ . Then  $\ell_J(x_n \oplus 1), \ell_J(x_n \oplus 1) \leq \ell_J(g_n)$  and  $\ell_J([x_n \oplus 1, y_n \oplus 1]) = \ell_J(g_n)$ . Hence if we pass to the ultraproduct  $\mathbf{x}, \mathbf{y}, [\mathbf{x}, \mathbf{y}] \in N$ .

Now assume  $g_n$  belongs to a symplectic, orthogonal or unitary group  $\mathrm{SI}_n(q)$ . As made clear above we are free to assume  $\ell_J(g_n) = \ell_r(g_n)$ . We use the geometric considerations in Subsection 3.1, especially Lemma 3.3. Let  $W := \ker(1 - g_n)$  and  $W'$  a complement of  $\mathrm{rad}(W)$  in  $W$ . We obtain a non-degenerate subspace  $U$  in  $\mathbb{F}_q^n$  such that  $g_n$  acts as the identity on  $U$ ,  $U^\perp = W'$  and  $\dim(U^\perp) \leq 2(n - \dim \ker(1 - g_n))$ . Thus we can restrict  $g_n$  to  $U^\perp$  and proceed as above.  $\square$

**3.17 Corollary** *Let  $\mathbf{G}$  be an ultraproduct of finite simple groups,  $\mathbf{g} \in \mathbf{G} \setminus \{1\}$  and  $N$  a proper normal subgroup in  $N(\mathbf{g})$ . Then every element in the group  $N(\mathbf{g})/N$  is a commutator.*

*Proof.* By the proposition  $\mathbf{g}$  is a commutator of some  $\mathbf{x}, \mathbf{y} \in N$ . Then  $\ell(g_n) = \ell([x_n, y_n]) \leq 2 \min(\ell(x_n), \ell(y_n))$  for  $\mathbf{u}$ -almost all  $n$  and therefore  $[\mathbf{x}, \mathbf{y}] \preceq \mathbf{x}, \mathbf{y}$  and  $\mathbf{g} \preceq \mathbf{x}, \mathbf{y}$ . If  $\mathbf{x}$  was in  $N$ , the contradiction  $N(\mathbf{g}) \subset N(\mathbf{x}) \subset N \subset N_0(\mathbf{g}) \subset N(\mathbf{g})$  would follow. Of course also  $\mathbf{y}, [\mathbf{x}, \mathbf{y}] \notin N$ . Hence every element in  $N(\mathbf{g})/N$  is a commutator.  $\square$

By the maximality of predecessors we deduce one more corollary.

**3.18 Corollary** *If  $\mathbf{G}$  is an ultraproduct of finite simple groups, then the group  $N(\mathbf{g})/N_0(\mathbf{g})$  is perfect and simple for all  $\mathbf{g} \in \mathbf{G} \setminus \{1\}$ .*

In order to prove the previous corollary, it is enough to assume in the proof of Proposition 3.16 that there exists a universal constant  $c$ , such that every element in the commutator subgroup of a non-abelian quasisimple group is a product of at most  $c$  commutators. This was established by Wilson [17] long before the Ore Conjecture was solved.

## 4 Ultraproducts of compact connected simple Lie groups

We want to show that an analogue of Theorem 3.9 holds for well behaved Lie groups.

### 4.1 Bounded generation in compact connected simple Lie groups

The motivation for this paragraph is taken from [13], Paragraph 5.5.4, wherefrom we freely cite all we need. The goal is to refine the methods from *ibid.* to obtain the result that in compact connected simple Lie groups an element which is not much longer, in a certain sense, than some other element, can be written as a bounded product of conjugates of the latter.

Let  $G$  be a compact connected simple Lie group. (That is a compact connected perfect Lie group, which is simple modulo its centre.) Then  $G$  contains a maximal compact connected abelian subgroup  $T$ , called a maximal torus of  $G$ . The dimension as a manifold of  $T$  is called the rank of  $G$ . The choice of a torus  $T$  is unique up to conjugation and every element in  $G$  is conjugate to one in  $T$ . Assume that the rank of  $G$  is  $r$ . Then there is a set  $\Phi = \{\beta_1, \dots, \beta_r\}$  of fundamental roots, determining  $T$ . Each root  $\alpha$  corresponds to a character  $\alpha : T \rightarrow S^1$ . Then

$$\bigcap_{i=1}^r \ker \beta_i = Z(G).$$

A complex number  $\mu$  in  $S^1$  can be written uniquely as  $e^{i\vartheta}$ , where  $\vartheta \in ]-\pi, \pi]$ . We call  $l(\mu) := |\vartheta|$  the **angle** of  $\mu$ . Now we define

$$\lambda(g) := \frac{1}{\pi r} \sum_{i=1}^r l(\beta_i(t))$$

for all  $g \in T$ . Proposition 5.11 in [13], the proof of which is spread over Subsection 5.5 *ibid.*, includes the following result.

**4.1 Proposition** *The function  $\lambda : T \rightarrow \mathbb{R}$  is an invariant pseudo length function on  $T$  and  $\lambda(g) = 0$  if and only if  $g \in Z(G)$ .*

From now on we will safely assume that  $G$  is a simply connected Lie group, since  $\lambda$  is zero on the center of  $G$  and thus well defined on the quotient  $G/Z(G)$ . We continue with further features of the internal structure of a (simply connected) compact connected simple Lie group, as outlined in [13]. We make adjustments to the text and notation of this reference when needed.

For every character  $\alpha$  we have a cocharacter  $\eta_\alpha : S^1 \rightarrow T$  such that  $\alpha(\eta_\alpha(\mu)) = \mu^2$  for all  $\mu \in S^1$ . For every pair of opposite roots  $\pm\alpha$  of  $\Phi$  there is a homomorphism  $\varphi_\alpha : \text{SU}(2) \rightarrow G$  such that  $\eta_\alpha$  is the restriction of  $\varphi_\alpha$  to the subgroup of diagonal matrices of  $\text{SU}(2)$ . For every root  $\alpha$  we define subgroups associated with it. There is  $T_\alpha := \{g \in T \mid \alpha(g) = 1\} \subset T$ . Then we have  $S_\alpha = S_{-\alpha}$ , the image of  $\text{SU}(2) \subset G$  under  $\varphi_\alpha$ .  $S_\alpha$  commutes elementwise with  $T_\alpha$  and  $T$  is contained in the central product  $S_\alpha T_\alpha$ . At last we define the one parameter torus  $H_\alpha$  as the image of the cocharacter  $\eta_\alpha$ . For fundamental roots  $\beta_i$  we use the self-explanatory shorthand notation  $T_i$ ,  $S_i$  and  $H_i$ . Then  $T$  equals the direct product  $H_1 H_2 \dots H_r$ .

Each central element  $g \in Z(G)$  can be decomposed into the product of commuting factors  $g = g_1 \dots g_r$ , where  $g_i \in H_i$ . We define

$$g'_i := g_1 \dots g_{i-1} g_{i+1} \dots g_r.$$

Then it is clear that  $g = g_i g'_i = g'_i g_i$  for any  $i$ . Moreover  $\lambda(g_i) = l(\beta_i(g_i)) = l(\beta(g))$  and hence  $\lambda(g) = \sum_{i=1}^r \lambda(g_i)$ . Also  $l(\beta_i(g'_i)) = 0$ , since  $g'_i \in T_i$ .

The following result can be found in the proof of Lemma 5.20 in [13].

**4.2 Lemma** *Let  $G = \text{SU}(2)$  and  $g, h$  be non-trivial elements in  $G$  such that  $\lambda(g) \leq m\lambda(h)$ ,  $m \geq 2$  an integer. Then  $g$  is a product of  $m$  conjugates of  $h$ .*

We give two lemmas concerning linear combinations of roots, and the resulting impact on the structure of  $G$ .

**4.3 Lemma** *In any simple root system  $\Phi$  every long root  $\beta$  can be written as  $\beta = \alpha_1 + \alpha_2$  for short roots  $\alpha_1$  and  $\alpha_2$ . Every short root  $\alpha$  can be written as  $\alpha = \mu\beta_1 + \mu\beta_2$ , where  $\mu \in \pm\{\frac{1}{3}, \frac{1}{2}, 1\}$ . These are the only coefficients that can appear in a linear combination of two roots to a third.*

**Proof.** The lemma follows from inspection of the standard representations of root systems.

If  $\Phi$  is of type  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$  or  $E_8$  all roots have the same length and there is nothing to prove. In case of  $\Phi$  being of type  $B_n$ , the roots are exactly the integer vectors  $v$  in  $\mathbb{R}^n$  with Euclidean norm  $|v| = 1$  or  $|v| = \sqrt{2}$ . For type  $C_n$  we have  $\Phi = \{v \in \mathbb{Z}^n \mid |v| = \sqrt{2}\} \cup \{v \in (2\mathbb{Z})^n \mid |v| = 2\}$ . We see that in these cases  $\mu = \pm\frac{1}{2}$ . If  $\Phi$  is of type  $F_4$ , it is the union of the set of all vectors in  $\mathbb{R}^4$  with two or one components equal to  $\pm 1$  and the others equal to 0 and the set of vectors with all components being  $\pm\frac{1}{2}$ . Here  $\mu$  is either  $\pm\frac{1}{2}$  or  $\pm 1$ , depending on the short root. In the remaining case of type  $G_2$  we represent  $\Phi$  by vectors in  $\mathbb{R}^3$ , the short roots being

$$(1, -1, 0), (-1, 1, 0), (1, 0, -1), (-1, 0, 1), (0, 1, -1), (0, -1, 1)$$

and the long roots

$$(2, -1, -1), (-2, 1, 1), (1, -2, 1), (-1, 2, -1), (1, 1, -2), (-1, -1, 2).$$

Again a close look implies the claim with  $\mu = \pm\frac{1}{3}$ .  $\square$

**4.4 Lemma** *Let  $\alpha, \beta$  be fundamental roots of different lengths and  $g$  an element in  $H_\alpha$  such that  $\lambda(g) = \varepsilon$ . Then there are elements  $w_1$  and  $w_2$  in the Weyl group such that  $H_\alpha \subset H_\beta^{w_1} H_\beta^{w_2}$  and in particular  $g$  equals the product  $g_1 g_2$ , where  $g_i \in H_\beta^{w_i}$  are elements of length  $\lambda(g_i) \leq \varepsilon$ .*

**Proof.** The inclusion  $H_\alpha \subset H_\beta^{w_1} H_\beta^{w_2}$  can be found in [13] in the proof of Lemma 5.19 and the argument goes as follows. The Weyl group  $W$  acts on the roots. There are elements  $w_1$  and  $w_2$  in  $W$  such that  $\alpha$  equals the linear combination  $\mu_1 \beta^{w_1} + \mu_2 \beta^{w_2}$ . The claim follows.

We have to go into detail and take care of lengths of elements in the product. To each root  $\delta$  corresponds a coroot  $h_\delta$ . Section 23 in [4] shows that there is a normalization of coroots such that we can assume  $h_{\delta+\zeta} = h_\delta + h_\zeta$ . The homomorphism  $\eta_\delta$  is induced by  $\vartheta \mapsto \exp(\vartheta i h_\delta)$ , where the length of an element in the image is  $\vartheta$  if  $\vartheta \in [0, \pi]$  and  $2\pi - \vartheta$  if  $\vartheta \in [\pi, 2\pi]$ . (Confer [16], Theorems 6.20, 4.8, 4.16.) From Lemma 4.3 we know that the coefficients  $\mu := \mu_1 = \mu_2$  are of absolute value not more than 1 and by reflection with elements in the Weyl group we can assume  $\mu$  positive. Hence

$$h_{\gamma_1} + h_{\gamma_2} = h_{\gamma_1 + \gamma_2} = h_{\mu^{-1}\alpha} = \mu^{-1} h_\alpha$$

and the coroots obey the same linear relation as the roots.

Assume without loss of generality  $g = \eta_\alpha(\varepsilon)$ . If we write  $\gamma_i := \beta^{w_i}$ ,  $i = 1, 2$ , then  $\alpha = \mu\gamma_1 + \mu\gamma_2$  implies

$$\eta_\alpha(\varepsilon) = \exp(\varepsilon i h_\alpha) = \exp(\mu \varepsilon i h_{\gamma_1}) \exp(\mu \varepsilon i h_{\gamma_2}) = \eta_{\gamma_1}(\mu \varepsilon) \eta_{\gamma_2}(\mu \varepsilon).$$

Hence  $g = \eta_\alpha(\varepsilon) \in H_\alpha$  is the product of elements  $g_i = \eta_{\gamma_i}(\mu\varepsilon)$  of length  $\mu\varepsilon \leq \varepsilon$  in the subgroups  $H_\beta^{w_1}$  and  $H_\beta^{w_2}$ .  $\square$

We are now ready to generalize Lemma 4.2 to an arbitrary compact connected simple Lie group  $G$ . We use the interplay of the groups  $S_i$  and the Weyl group  $W$ , and the decomposition of the maximal torus  $T$  into subgroups  $H_i$ .

**4.5 Lemma** *Let  $g_i \in H_i$  and  $h_j \in H_j$ , corresponding to  $g$  and  $h$  in  $T$ , such that  $l(\beta_i(g)) \leq ml(\beta_j(h))$ , where  $m$  is an even integer. Then  $g_i \in (h^G \cup h^{-G})^{4m}$ .*

*Proof.* The proof splits in two cases whether  $\beta_i$  and  $\beta_j$  have the same length or not.

If  $\beta_i$  and  $\beta_j$  are roots of the same length, then there is an element  $v$  in the Weyl group  $W$  such that  $H_i^v = H_j$ . We see that this entails  $g_i^v \in H_j \subset S_j$  and, because we used conjugation,  $l(\beta_j(g_i^v)) = \frac{1}{\pi r} \lambda(g_i^v) = \frac{1}{\pi r} \lambda(g_i) = l(\beta_i(g_i))$ . Now  $g_i^v \in (h_j^{S_j})^m$  by Lemma 4.2.

We compute

$$(h^{S_j})^m = ((h_j h_j')^{S_j})^m = (h_j^{S_j})^m \cdot h_j'^m$$

to deduce

$$g_i \in ((h^{S_j})^m h_j'^{-m})^{v^{-1}}.$$

Now  $l(\beta_j(1)) = 0$  and by Lemma 4.2,  $1 \in (h_j^{S_j})^2$ . Therefore

$$h_j'^2 \in h_1^2 \cdot \dots \cdot h_{j-1}^2 \cdot (h_j^{S_j})^2 \cdot h_{j+1}^2 \cdot \dots \cdot h_r^2 = (h^{S_j})^2,$$

because  $h_i$  commutes with every element in  $S_j$ , whenever  $i \neq j$ . Note that this works equally well for  $h^{-1}$  instead of  $h$ . Because we assumed  $m$  even, we arrive at

$$g_i \in ((h^{S_j})^m (h^{-S_j})^m)^{v^{-1}} \subset (h^G \cup h^{-G})^{2m}.$$

If  $\beta_i$  and  $\beta_j$  are roots of different lengths, Lemma 4.4 gives the existence of elements  $w_1$  and  $w_2$  such that  $g_i = f_1^{w_1} f_2^{w_2}$ , where  $f_k$  is in  $H_j$  and  $\lambda(f_k) \leq \lambda(g_i)$ , for  $k = 1, 2$ . Then, again by Lemma 4.2,  $f_k \in (h_j^{S_j})^m$ , and we obtain

$$g_i \in ((h_j^{S_j})^m)^{w_1} \cdot ((h_j^{S_j})^m)^{w_2}.$$

We now proceed as above to deduce

$$g_i \in ((h^{S_j})^m (h^{-S_j})^m)^{w_1} \cdot ((h^{S_j})^m (h^{-S_j})^m)^{w_2} \subset (h^G \cup h^{-G})^{4m}. \quad \square$$

The next lemma is modelled after Case 1 in Lemma 5.19 in [13].

**4.6 Theorem** *Let  $\varepsilon > 0$  and  $G$  be a compact connected simple Lie group of rank  $r$ . Assume  $g$  and  $h$  are non-trivial elements in  $T$  satisfying  $\lambda(h) = \varepsilon$  and  $\lambda(g) \leq m\lambda(h)$  for an even integer number  $m$ . Then*

$$g \in (h^G \cup h^{-G})^{4m+8r}.$$

Proof. Write  $g = g_1 \cdot \dots \cdot g_r$ ,  $h = h_1 \cdot \dots \cdot h_r$ , where  $g_i, h_i \in H_i$ . For reasons of averaging there is one fundamental root  $\beta_j$  such that  $l(\beta_j(h)) \geq \varepsilon\pi$ . Let  $m_i \geq 2$  be the smallest even integer such that  $\lambda(g_i) \leq m_i \lambda(h_j)$ . Then  $m_i$  cannot be more than  $m$  for any  $i$ . In the worst case, there might be many  $m_i$  which are very small (such as  $m_i = 2$ ), while some  $m_i$  is close to  $m$ . Therefore we must estimate  $\sum_{i=1}^r m_i \leq m + 2r$ . We now use Lemma 4.5 to obtain for all  $i$

$$g_i \in (h^G \cup h^{-G})^{4m_i},$$

independently of the length of roots involved. Because  $g$  is the product of the  $g_i$ , and summing the  $m_i$  gives at most  $m + 2r$ ,  $g$  is a product of  $4m + 8r$  or less conjugates of  $h$  and  $h^{-1}$ .  $\square$

## 4.2 Normal subgroups of ultraproducts of compact connected simple Lie groups

**4.7 Proposition** *Let  $\|u\|$  denote the normalized  $l^1$ -norm on the group of unitary complex matrices  $U_n$ . Then  $\ell_1(u) := \frac{1}{2}\|1 - u\|$  defines an invariant length function.*

Proof. This follows from well known properties of unitary groups and matrix norms.  $\square$

**4.8 Lemma** *Let  $G$  be compact connected simple Lie group, which we imagine as embedded in  $U_n$ . Then the induced length function  $\ell_1$  is independent of the chosen embedding into  $U_n$ . For any  $g \in T$*

$$\ell_1(g) \leq \lambda(g) \leq 2\ell_1(g).$$

Proof. We can write  $\ell_1(g) = \frac{1}{2n} \sum_{i=1}^n |1 - \mu_i|$ , where  $\mu_i$  are the eigenvalues of  $g$ . By definition the maximal torus  $T$  of  $G$  is the intersection of a maximal torus  $T_n$  of  $U_n$  with  $G$ . By conjugation we can assume without loss of generality that  $T_n$  is the torus of diagonal matrices. It follows that every  $g$  in  $G$  has not more than  $r$  eigenvalues different from 1, since  $g$  can be conjugated to a torus element, which is unique up to the order of the diagonal entries. We see that this implies also the independence of  $\ell_1$  of the chosen embedding. Now the stated inequality follows from geometric considerations.  $\square$

**4.9 Theorem** *Let  $G_n$  be compact connected simple Lie groups. If  $\mathbf{g} \in \mathbf{G} := \prod_u G_n$  satisfies  $\lambda(\mathbf{g}) > 0$ , then  $N(\mathbf{g}) = \mathbf{G}$ . The set  $\mathbf{N}$  of all  $\mathbf{g}$  such that  $\lambda(\mathbf{g}) = 0$  is a normal subgroup and  $\mathbf{G}/\mathbf{N}$  is simple.*

Proof. We can assume  $g_n \in T_n$ , where  $T_n$  is a maximal torus of  $G_n$ . Then the first part of the theorem follows already from Lemma 5.19 in [13]. For groups of bounded rank we can alternatively use Theorem 4.6.

By Lemma 4.8  $\lambda(\mathbf{g}) = 0$  is equivalent to  $\ell_1(\mathbf{g}) = 0$ . Because  $\ell_1$  is a length function,  $\mathbf{N}$  is a normal subgroup. From the first part of the theorem we deduce that  $\mathbf{G}/\mathbf{N}$  is simple.  $\square$

We define  $\mathbf{g} \preceq \mathbf{h}$  for  $\mathbf{g}, \mathbf{h} \in \mathbf{G} \setminus \{1\}$  as in Paragraph 3.2, except that we use  $\ell_1$  as our length of choice. Then Lemma 3.10 immediately implies  $\mathbf{g} \preceq \mathbf{h}$  whenever  $\mathbf{g} \in N(\mathbf{h})$ .



**4.10 Lemma** *Let  $\mathbf{G}$  be an ultraproduct of compact connected simple Lie groups  $G_n$  of bounded rank and assume  $\mathbf{g} \preceq \mathbf{h}$  for non-trivial elements  $\mathbf{g}$  and  $\mathbf{h}$  in  $\mathbf{G}$ . Then  $\mathbf{g} \in N(\mathbf{h})$ .*

*Proof.* The hypothesis assures  $\lambda(g_n) \leq m\lambda(h_n)$  for almost all  $n$  and a suitable constant  $m$ . Following Theorem 4.6 we immediately obtain  $g_n \in C(h_n^{\pm 1})^{4m+8r}$ , where  $r$  is the bound on the rank of the groups  $G_n$ . Hence

$$\mathbf{g} \in C(\mathbf{h}^{\pm 1})^{4m+8r} \subset N(\mathbf{h}) \quad \square$$

We are now ready to prove the analogue of Theorem 3.9 for Lie groups of bounded rank.

**4.11 Theorem** *Let  $G_n$  be compact connected simple Lie groups of bounded rank. Then the set  $\mathfrak{N}$  of normal subgroups of  $\mathbf{G} := \prod_{\mathbf{u}} G_n$  is linearly ordered by inclusion.*

*Proof.* Exactly like in the proof of Theorem 3.9 we show that the set  $\mathfrak{N}_0$  of normal closures of elements of  $\mathbf{G}$  is order isomorphic to a subset of  $\mathbf{K}/\equiv$ . This is the quotient of  $\mathbf{K} := \prod_{\mathbf{u}} [0, 1]$  by the equivalence relation  $\equiv$ , which defines  $\mathbf{a}$  and  $\mathbf{b}$  equivalent if

$$0 < \lim_{\mathbf{u}} \frac{a_n}{b_n} < \infty.$$

Because a maximal torus  $T$  in a Lie group of rank  $r$  is isomorphic to the standard torus  $(S^1)^r$ , it is clear that for any prescribed  $a$  in  $[0, 1]$  there is an element in  $T$  with length  $a$ . Hence  $\mathfrak{N}_0$  is isomorphic to  $\mathbf{K}/\equiv$ . Now an application of Lemma 3.14 shows that also  $\mathfrak{N}$  is linearly ordered.  $\square$

Unfortunately, unlike for finite simple groups, the theorem turns out to be false if there is no bound on the rank. We illustrate this fact as follows.

Let  $G_n := U_n = U_n(\mathbb{C})$  for  $n = 2k$ . We consider elements

$$g_n = \text{diag}(e^{i\pi/n^2}, \dots, e^{i\pi/n^2}, 1, \dots, 1), \quad h_n = \text{diag}(-1, 1, \dots, 1)$$

in the maximal torus  $T$  of diagonal matrices.

If we assume that  $\mathbf{g} \in N(\mathbf{h})$ , then Lemma 3.10 implies the existence of a constant  $m$  such that  $\ell_r(g_n) \leq m\ell_r(h_n)$  for infinitely many  $n$ . But obviously the left hand side equals  $\frac{1}{2}$  and the right hand side equals  $\frac{m}{n}$ , which does not fit together well. On the other hand  $\mathbf{h} \in N(\mathbf{g})$  would imply

$$\lambda(h_n) \leq 2\ell_1(h_n) \leq 2m\ell_1(g_n) \leq 2m\lambda(g_n),$$

for some (other) constant  $m$  and infinitely many  $n$ . We evaluate  $\lambda(g_n) = \frac{1}{2n^2}$  and  $\lambda(h_n) = \frac{1}{n}$  to obtain a contradiction once again, and must conclude that neither  $N(\mathbf{g}) \subset N(\mathbf{h})$  nor  $N(\mathbf{h}) \subset N(\mathbf{g})$  holds. Therefore the normal subgroups in the ultraproduct of (projective) unitary groups cannot be linearly ordered.

Despite this setback we try to see how far we can get. Let  $g$  be an element in the maximal torus of a compact connected simple Lie group of rank  $n$ . We define a function  $F_g : \mathbb{N} \rightarrow [0, 1]$  by

$$F_g(i) := \begin{cases} \frac{1}{2}|1 - \beta_{\sigma(i)}(g)|, & i \in [n], \\ 0, & i > n, \end{cases}$$

where  $\sigma$  is a permutation of  $[n]$  such that  $F_g(i) \geq F_g(i+1)$  results for all  $i \geq 1$ . Let  $\mathbf{F}$  and  $\mathbf{H}$  be representatives of sequences  $(F_n)$  and  $(H_n)$ , respectively, of functions in the ultraproduct

$$\mathbf{M} := \prod_{\mathbf{u}} \mathcal{F}_n(\mathbb{N}, [0, 1]),$$

where  $\mathcal{F}_n(\mathbb{N}, [0, 1])$  is the set of decreasing functions  $\mathbb{N} \rightarrow [0, 1]$  with support contained in  $[n]$ . We let  $\mathbf{F} \preceq \mathbf{H}$  if and only if there are constants  $c$  and  $k \in \mathbb{N}$  such that for  $\mathbf{u}$ -almost all  $n$

$$F_n(ki + 1) \leq cH_n(i + 1),$$

whenever  $i \geq 0$ . It is clear that this defines a quasiorder on the space  $\mathbf{M}$ . We let  $\mathbf{F} \equiv \mathbf{H}$  if  $\mathbf{F} \preceq \mathbf{H}$  and  $\mathbf{H} \preceq \mathbf{F}$  to obtain the quotient space  $\mathbf{M}/\equiv$  with the induced ordering.

If  $\mathbf{g} \in \mathbf{G} \setminus \{1\}$  we define  $F_{\mathbf{g}}$  as the element in  $\mathbf{M}$  associated with  $(F_{g_n})_n$ . With these two notions at hand let  $\mathbf{g} \preceq \mathbf{h}$  be equivalent to  $F_{\mathbf{g}} \preceq F_{\mathbf{h}}$ .

**4.12 Lemma** *Let  $g$  and  $h$  be elements in  $U_n$ . Then, if  $i, j \geq 0$ .*

$$F_{gh}(i + j + 1) \leq 4(F_g(i + 1) + F_h(j + 1)).$$

*Proof.* We begin by comparing  $F_g(i)$  with the  $i$ -th singular value  $s_i(1 - g)$  of  $(1 - g)$ . The latter is defined as

$$s_i(1 - g) = \sqrt{\lambda_i((1 - g)^*(1 - g))},$$

where  $\lambda_i((1 - g)^*(1 - g))$  is some eigenvalue of  $(1 - g)^*(1 - g)$ , and we assume the  $s_i(1 - g)$  in decreasing order with respect to their absolute value. Now the eigenvalues of  $g$  are of the form  $e^{i\vartheta}$ , so that  $s_i(1 - g) = \sqrt{2(1 - \operatorname{Re} e^{i\vartheta})}$ . Hence for geometric reasons

$$\frac{1}{2}F_g(i) \leq s_i(1 - g) \leq 2F_g(i).$$

If  $i, j \geq 0$  and  $i + j + 1 \leq n$  we have by the Ky Fan singular value inequality

$$\begin{aligned} s_{i+j+1}(1 - gh) &= s_{i+j}((1 - g)h + (1 - h)) \\ &\leq s_{i+1}((1 - g)h) + s_{j+1}(1 - h) \\ &= s_{i+1}(1 - g) + s_{j+1}(1 - h). \end{aligned}$$

(Confer [8].) This implies

$$F_{gh}(i + j + 1) \leq 4(F_g(i + 1) + F_h(j + 1)),$$

as long as  $i, j \geq 0$ . □

**4.13 Proposition** *Let  $\mathbf{g}$  and  $\mathbf{h}$ , both not equal to 1, be elements in an ultraproduct of compact connected simple Lie group  $G_n$  such that  $\mathbf{g} \in N(\mathbf{h})$ . Then  $\mathbf{g} \preceq \mathbf{h}$ .*

**Proof.** We assume that  $g$  is a product of  $k$  conjugates of  $h^{\pm 1}$ . This implies that  $g_n \in G_n$  is a product of  $k$  conjugates of  $h_n^{\pm 1}$  for  $u$ -almost all  $n$ . By conjugating we can assume  $g_n$  and  $h_n$  in a maximal torus of  $G_n$ . We only have to take care of  $n$  sufficiently larger than  $k$ . Without loss of generality we assume that  $G_n$  has rank  $n$  and imagine  $G_n$  embedded in a unitary group, in order to use Lemma 4.12. In a group of such a large rank now for  $i \geq 0$

$$F_g(ki + 1) \leq 4^k k F_h(i + 1)$$

holds, because  $F_h$  is invariant under conjugation of  $h$  with unitaries.  $\square$

**4.14 Lemma** *There is a natural number  $s \geq 3$  such that the following holds. Let  $\sigma$  be a permutation of the numbers  $[n]$ , where  $n$  is divisible by  $s$  and consider the vector  $s = (\sigma(1), \sigma(2), \dots, \sigma(n))$ . Then one can partition  $s$  into  $s$  vectors  $v_i := (a_{i,1}, \dots, a_{i,n/s})$  such that  $|a_{i,j} - a_{i,k}| \neq 1$  for all  $i = 1 \dots s$  and  $j, k = 1 \dots n/s$ , and  $a_{i,j} = \sigma(k)$  implies  $|sj - k| \leq s - 1$ .*

**Proof.** We reformulate the problem in graph theoretical terms. Consider the vector  $C := (1, 2, \dots, n)$  as a graph, where  $i, j$  are connected if  $|i - j| \in \{1, n - 1\}$ , i.e. the cycle  $C_n$ . We assign a vertex  $i$  of this graph the label  $\lfloor \frac{\sigma^{-1}(i) + s - 1}{s} \rfloor$ . Thus we use  $n/s$  different labels, each one occuring exactly  $s$  times. If  $s \geq s\chi(C_n)$ , where  $s\chi(C_n)$  denotes the strong coloring number of  $C_n$ , then there is a proper coloring of  $C_n$  such that no two vertices with the same label have the same color. Now let  $v_i$  be the vector of vertices of color  $i$ , in the ordering prescribed by the labels. Then it follows immediately that no two consecutive numbers appear in the same  $v_i$ . If  $a_{i,j} = \sigma(k)$ , then  $j = \lfloor \frac{k + s - 1}{s} \rfloor$  and the difference  $|sj - k|$  is strictly less than  $s$ .

Since it is known that the strong coloring number of  $C_n$  can be bounded independently of  $n$ , the claim follows.  $\square$

Note that the constant  $s\chi(C_n)$  in the previous lemma can be made explicit. Alon in [2] mentions the bound of  $s\chi(C_n) \leq 4$  (for  $n$  divisible by 4), credited to de la Vega, Fellows and himself. The usual proofs invoke the probabilistic methods such as the Lovász local lemma. Fleischner and Stiebnitz proved  $s\chi(C_n) = 3$  and there is an elementary proof, presented by Sachs in [15].

**4.15 Lemma** *Let  $G$  be a compact connected simple Lie group, the rank  $r$  of which exceeds both 10 and  $8k$ , for a natural number  $k$ . Assume  $g$  and  $h$  are non-trivial elements in the maximal torus  $T$  satisfying  $F_g(ki + 1) \leq m F_h(i + 1)$  if  $i \geq 0$ ,  $m \in \mathbb{N}$  an even integer. Then*

$$g \in (h^G \cup h^{-G})^{384km^3 + 8(4k+1)m}.$$

**Proof.** Considering the rank requirements,  $G$  is a classical group of type  $A_r$ ,  $B_r$ ,  $C_r$  or  $D_r$ . Then without loss of generality the roots  $\beta_i$ ,  $i = 1 \dots r - 1$  form a root system of type  $A_{r-1}$  and the root  $\beta_r$  possibly has a different length. Roots  $\beta_j$  and  $\beta_i$  are orthogonal, whenever  $|i - j| \geq 2$  and we will say that  $i$  is orthogonal to  $j$  in that case.

With  $N$  the smallest natural number divisible by 3 such that  $Nk \leq r - k - 1$ , we define  $N$ -tuples  $A_l := (l, k + l, 2k + l, \dots, Nk + l)$  for  $l = 1 \dots k$  and  $A_0 :=$

$(1, 2, \dots, N)$ . We choose the permutation  $\sigma$  implicitly by writing  $F_g(i) = \frac{1}{2}|1 - \beta_{\sigma(i)}(g)|$  like above. Likewise we have  $\tau$  corresponding to  $h$ . Both permutations act coordinatewise on  $N$ -tuples. If we chose  $i \geq 0$ , then

$$\frac{1}{2}|1 - \beta_{\tau(ki+1)}(g)| = F_g(ki + l) \leq mF_h(i + 1) = \frac{1}{2}m|1 - \beta_{\sigma(i+1)}(h)|.$$

If  $l \in \{1, \dots, k\}$  we hence obtain

$$l(\beta_{\tau(ki+l)}(g)) \leq 4ml(\beta_{\tau(i+1)}(h)).$$

Without loss of generality, we can assume the worst case that  $A_k$  contains  $N$  consecutive numbers. Then by Lemma 4.14 and the subsequent remarks there is a partition of  $\tau(A_0)$  into tuples  $B_1, B_2$  and  $B_3$  with the same number of elements such that the entries in  $B_i$  are pairwise orthogonal for  $i = 1, 2, 3$ . In the same way we obtain  $C_{i,l}$ ,  $i = 1, 2, 3$ , from the sets  $\sigma(A_l)$ ,  $l = 1 \dots k$ . By Lemma 5.21 in [13] (here we use orthogonality) there are elements  $w_{i,l}$  in the Weyl group of  $G$  that map the vectors of fundamental roots corresponding to  $B_i$  to the vectors of fundamental roots corresponding to  $C_{i,l}$  for all  $i = 1, 2, 3$ ,  $l = 1 \dots k$ . Then the argument of Lemma 4.5, applied simultaneously to all  $g_i$ ,  $i \in C_{j,l}$ , yields  $\prod_{i \in C_{j,l}} g_i \in (h^G \cup h^{-G})^{2 \cdot (4m)^3}$ , where the exponent 3 is derived from the fact that Lemma 4.14 with  $s = 3$  guarantees that indices are at distance at most 2 from their optimal position. We can use the factor 2, because all roots under consideration have the same length. When we reconstruct most of  $g$  in this way, we arrive at

$$\prod_{i \in \bigcup C_{j,l}} g_i \in (h^G \cup h^{-G})^{128m^3 \cdot 3k}.$$

What remains are the indices left out in the above procedure. The number of these is  $r - Nk \leq 4k$  by choice of  $N$ . If  $i \leq r - 1$ , using  $h_{\tau(1)}$ , we can generate the  $g_i$  separately in  $2 \cdot 4m$  steps like in Lemma 4.5. The last root  $\beta_r$  possibly requires the second argument in the proof of Lemma 4.5, which results in adding  $4 \cdot 4m$ . Hence generating the missing parts of  $g$  can be done in  $(4k - 1) \cdot 8m + 16m = 8(4k + 1)m$  steps.

All in all we end up with

$$g \in (h^G \cup h^{-G})^{384km^3 + 8(4k+1)m}$$

as claimed.  $\square$

**4.16 Theorem** *Let  $\mathbf{g}$  and  $\mathbf{h}$  be elements in the ultraproduct  $\mathbf{G}$  of compact connected simple Lie groups of unbounded rank. Then  $\mathbf{g} \preceq \mathbf{h}$  is equivalent to  $\mathbf{g} \in N(\mathbf{h})$ .*

*Proof.* The first implication was already proved in Proposition 4.13. The proof of the second is an application of Lemma 4.15, analogous to the proofs of Theorem 3.9 or Theorem 4.11.  $\square$

Up to now it is clear that the set of normal closures of elements in  $\mathbf{G}$  is order isomorphic to  $\mathbf{M}/\equiv$ . What remains to be clarified is the influence of this ordering on the ordering of normal subgroups.

### 4.3 The lattice of normal subgroups

We are interested in the lattice of normal subgroups of groups  $G$ . The lattice operations are  $N \wedge M = N \cap M$  and  $N \vee M = NM$ , the normal subgroup generated by  $N$  and  $M$ , for any choice of normal subgroups in  $G$ . It is well known that the lattice of normal subgroups of any group is modular, that is for normal subgroups  $L, M, N$  the modular law

$$((L \wedge N) \vee M) \wedge N = (L \wedge N) \vee (M \wedge N)$$

holds.

**4.17 Lemma** *Let  $\mathbf{G}$  be an ultraproduct of compact connected simple Lie groups and  $\mathbf{g}, \mathbf{h}$  in a maximal torus  $\mathbf{T}$  of  $\mathbf{G}$ . Then there are  $\mathbf{a}, \mathbf{b} \in \mathbf{T}$  such that  $N(\mathbf{g}) \wedge N(\mathbf{h}) = N(\mathbf{a})$  and  $N(\mathbf{g}) \vee N(\mathbf{h}) = N(\mathbf{b})$ .*

*Proof.* We define functions  $\mathbf{A} := \min(F_{\mathbf{g}}, F_{\mathbf{h}})$  and  $\mathbf{B} := \max(F_{\mathbf{g}}, F_{\mathbf{h}})$ . The plan is to show that there are actually elements  $\mathbf{a}$  and  $\mathbf{b}$  such that  $\mathbf{A} = F_{\mathbf{a}}$  and  $\mathbf{B} = F_{\mathbf{b}}$ . For some  $n$  consider the functions  $A_n := \min(F_{g_n}, F_{h_n})$ ,  $B_n := \min(k \mapsto 1, \max(F_{g_n}, F_{h_n}))$ . Let  $T_n$  be a maximal torus in the group  $G_n$  of rank  $r$ , where we can assume  $g_n, h_n \in T_n$ . Because  $T_n$  is isomorphic to  $(S^1)^r$  we find elements  $a_n$  and  $b_n$  in  $T_n$  such that  $F_{a_n} = A_n$  and  $F_{b_n} = B_n$ . This yields  $\mathbf{a}$  and  $\mathbf{b}$  as claimed.  $\square$

**4.18 Proposition** *Let  $\mathbf{G}$  be an ultraproduct of compact connected simple Lie groups. Then the set  $\mathfrak{N}_0$  of normal closures of elements in  $\mathbf{G} \setminus \{1\}$  is a distributive lattice.*

*Proof.* We already know that  $\mathfrak{N}_0$  is order isomorphic to  $\mathbf{M}/\equiv$ . It is clear that the latter is a distributive lattice with meet and join induced by the operations  $\min$  and  $\max$  applied to functions. Lemma 4.17 shows that the corresponding operations in  $\mathfrak{N}_0$  produce normal closures again.  $\square$

**4.19 Lemma** *Let  $G$  be a group. If the set of normal closures of elements in  $G$  is a distributive lattice, then the lattice of normal subgroups is distributive, too.*

*Proof.* Let  $L, M, N$  be any normal subgroups in  $G$ . We have to show that

$$(L \vee M) \wedge N = (L \wedge N) \vee (M \wedge N)$$

holds. Here the inclusion of the right hand side in the left hand side is true in general. Moreover by assumption the whole equation holds for normal closures of elements in  $G$ . Consider  $x \in (L \vee M) \wedge N$ . Then  $x \in L \vee M$  and  $x \in N$  because the meet operation is intersection of sets. Because the normal closure of  $L$  and  $M$  is the normal subgroup  $LM$ , there are  $a \in L$  and  $b \in M$  such that  $x$  equals the product  $ab$ . This means that  $x \in N(a) \vee N(b)$ . We also observe  $N(x) \subset N$  to obtain

$$\begin{aligned} x &\in (N(a) \vee N(b)) \wedge N(x) \\ &= (N(a) \wedge N(x)) \vee (N(b) \wedge N(x)) \\ &\subset (L \wedge N) \vee (M \wedge N). \end{aligned}$$

Thus the claim follows.  $\square$

The observations made in Proposition 4.18 and Lemma 4.17 suffice to prove the following result.

**4.20 Theorem** *If  $\mathbf{G}$  is an ultraproduct of compact connected simple Lie groups, then the lattice of normal subgroups of  $\mathbf{G}$  is distributive.*

## 5 Conclusion

We considered ultraproducts of finite simple groups and compact connected simple Lie groups. As a consequence of the Peter-Weyl Theorem, any compact simple group belongs to one of the two categories. We have to deal with the subcases of groups of bounded and unbounded rank, respectively, because the two behave differently as shown above. If we have an ultraproduct  $\mathbf{G}$  of compact simple groups the ultrafilter selects one kind of groups among the four listed possibilities, which determine the properties of  $\mathbf{G}$ . We will say that  $\mathbf{G}$  is of **bounded finite type**, **unbounded finite type**, **bounded Lie type** or **unbounded Lie type** if  $\mathbf{G}$  is essentially an ultraproduct of finite simple groups of bounded or unbounded rank or Lie groups of bounded or unbounded rank, respectively.

Recall the situation in the case of finite simple groups. We defined  $\mathbf{g} \preceq \mathbf{h}$  if

$$\lim_u \frac{\ell(g_n)}{\ell(h_n)} < \infty,$$

where  $\ell$  was one of the length functions  $\ell_r$  and  $\ell_J$ . For  $g \neq 1$  in a finite simple group of rank  $n$  define

$$F_g(k) := \begin{cases} 0 & \text{otherwise,} \\ 1 & \text{if } k \leq n\ell(g). \end{cases}$$

Then it is an elementary observation that  $F_g \preceq F_h$ , if and only if  $\mathbf{g} \preceq \mathbf{h}$  for non-trivial  $\mathbf{g}, \mathbf{h} \in \mathbf{G}$ . Using this last remark we can summarize our results in the following theorem.

**5.1 Theorem (Main Theorem)** *Let  $\mathbf{G}$  be an ultraproduct of non-abelian compact simple groups  $G_n$ . Let  $\mathbf{M}$  be the ultraproduct of sequences of decreasing functions  $F_n : \mathbb{N} \rightarrow [0, 1]$  with support of size less or equal to the rank of  $G_n$ . Define  $\mathbf{F} \preceq \mathbf{H}$  if there are constants  $c, k$  such that  $F_n(ki + 1) \leq cH_n(i + 1)$  for all  $i \geq 0$   $u$ -almost everywhere, and  $\mathbf{F} \equiv \mathbf{H}$  if  $\mathbf{F} \preceq \mathbf{H}$  as well as  $\mathbf{H} \preceq \mathbf{F}$ .*

1. *If  $\mathbf{G}$  is of unbounded Lie type, then the set of normal closures  $\mathfrak{N}_0$  of elements in  $\mathbf{G} \setminus \{1\}$  is a lattice isomorphic to the distributive lattice  $\mathbf{M}/\equiv$ . The lattice  $\mathfrak{N}$  of normal subgroups of  $\mathbf{G}$  is distributive.*
2. *If  $\mathbf{G}$  is of bounded Lie type, then  $\mathfrak{N}_0$  is isomorphic to the linearly ordered sublattice of  $\mathbf{M}/\equiv$  induced by the functions of bounded support and  $\mathfrak{N}$  is linearly ordered.*
3. *If  $\mathbf{G}$  is of unbounded finite type, then  $\mathfrak{N}_0$  is isomorphic to the linearly ordered sublattice of  $\mathbf{M}/\equiv$  induced by the functions  $F : \mathbb{N} \rightarrow \{0, 1\}$ . Again,  $\mathfrak{N}$  is linearly ordered.*

4. If  $\mathbf{G}$  is of bounded finite type, then  $\mathbf{G}$  is simple and  $\mathfrak{N}$  is isomorphic to the lattice 2.

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